A FRAMEWORK TO UNDERSTAND SMOOTHING AND GUARANTEES IN SAVINGS PRODUCTS

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ABSTRACT

We study the liabilities of with-profits products that arise when their returns are based on the linear combinations of a risky assets and cash returns. These liabilities are defined by an accrual rule that has strong similarities to the logic that defines: smoothing liability returns from the reserve, Constant Proportion Portfolio Insurance, bonuses in with-profits funds, and dynamic hedging of derivatives. By considering the accrual rule with parameters, we discover what the range of performance is, relative to the risky asset. Thus we identify accrual rules that outperform when the asset underperforms and vice versa. We simulate the behaviour of liabilities, for a wide range of parameters, over a long time horizon with, and without, return guarantees. Our analysis clearly identifies which accrual rules are optimal in respect of highest return, Sharpe ratio, and lowest risk to the policyholder or insurer.

KEYWORDS

Smoothing liability returns; liability return guarantees; asset liability mismatch; insurance guarantee reserve; dynamic asset allocation

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1. INTRODUCTION

In this paper, we consider a simple comparative framework to specify and analyse the performance of smoothing and guarantees of a savings account or product. The simplest guaranteed savings mechanism could be to invest in a short-term bond and re-invest interest earned. Although the outcome of this strategy is not certain, it is clearly capital guaranteed (as far as market risks are concerned). Many products exist that offer participation in the risky, but hopefully higher, returns of equity markets. However, due to the risk preferences of policyholders a guarantee is sold to limit their downside risk and even guarantee minimum vesting returns.

Financial management tools for these products range from offsetting bespoke derivative contracts and dynamic hedging on the one end, to mutual funds that earn returns linked to returns of an investment portfolio. In the latter case, the investment portfolio could allow asset allocation of risky assets to the extent that the total value of assets exceeds the book value of the liability. The rationale for dynamic asset allocation is to provide some protection against market downturns and possibly provide a minimum rate of return to the policyholders of the mutual fund. In addition to dynamic asset allocation, a discretionary bonus is added based on returns in excess of the guaranteed minimum rate and fund solvency.

The issues pertaining to the determination of discretionary return, and asset allocation as a means to manage the risk (created by the guarantee), are well known. For example, there is a substantial element of model risk in that the payoff is not explicitly known and products are long dated. In practice, there are also legal and commercial considerations. See for example Dippenaar et al. (2007).

Recent research on valuation and risk management focuses on modelling the fair market value of the liability using risk neutral pricing, e.g. Hibbert and Turnbull (2003) and Mahayni and Schlogle (2008). In this paper, we do not focus on the valuation in particular, but rather explore the possible behaviour of the liability as defined by specific accrual logic (i.e. through various stochastic differential equations). To this end we explore the effect of simple dynamic asset allocation driven by the ratio between assets and liabilities. We then explore the parametric family of all such strategies to identify important characteristics for policyholders.

2. ORGANISATION AND NOTATION

The rest of the paper is organised as follows. In section 3 we explore various ideas and concepts in a continuous time setting, thus creating a framework which is subsequently used to specify participation and guaranteed minimal returns. Concepts from the insurance world, dynamic asset allocation, and portfolio insurance are introduced before considering payoffs from derivatives. In the insurance context we introduce reserve smoothing, bonus formulae and asset allocation. In terms of dynamic asset allocation, we adapt Constant Proportion Portfolio Investment (CPPI) and define Constant Proportion Liability Investment (CPLI). For this strategy, and the pure reserve smoothing case, closed form solutions of the Stochastic Differential

Equations (SDEs) are given. For derivatives, we point to structures that are related to with-profits exposure and we end section 3 with a diagrammatic summary between different concepts.

In section 4, we move to a discrete time setting where the liability value is updated periodically. We consider a class of liabilities, possibly with vesting guarantees, where the portfolio weightings are linear functions in the ratio of assets to liabilities with suitable upper and lower bounds. The purpose of this section is to explore how parameter choices affect the return and volatility of the liability in comparison to the risky asset.

We draw a couple of conclusions from the work in section 5.

Throughout, we use the following conventions in respect of notation. The reference assets are denoted by A. The assets are invested in a risky asset class with index price S and a risk-free cash asset with balance B. Throughout we assume that the risky asset follows Geometric Brownian Motion:

$$dS = S(\mu\, dt + \sigma\, dZ),$$

where: μ denotes the risky return per year and dZ represents a standard Brownian motion. Rates are annual unless stated otherwise. A portion of the liability is seen to accrue at the risk-free rate r; thus for the risk-free asset B we assume $dB=rBdt$. The liabilities are denoted by L and if with guarantee, by G. Subscripts to A, S, B, and L indicate time dependency.

The return of assets are of course related to investments in risky and risk-free assets through portfolio weightings. In other words, we may write

$$\frac{dA}{A} = \omega \frac{dS}{S} + (1-\omega)\frac{dB}{B},$$

where ω denotes the percentage risky assets in the portfolio. We may represent the liability's return in continuous time as

$$\frac{dL}{L} = \xi \frac{dA}{A} + g dt,$$

for some function ξ and guaranteed rate g. The function ξ represents the process of determining the bonus in a business sense. By combining the returns of the assets, and the logic that defines the change in liability, we can write the liability in a more standard form.

By substituting for A, we write $\frac{dL}{L} = \xi(\omega \frac{dS}{S} + (1-\omega)\frac{dB}{B}) + g dt$, which states that the return of a liability could be seen as the weighted sum of the returns of risky, and risk-free (since $g dt = \frac{g}{r}\frac{dB}{B}$) assets. By choosing weighting functions

$\theta = \xi\omega$ and $\phi = \xi(1-\omega) + \frac{g}{r}$ we may define the liability return in a standard form

$$\frac{dL}{L} = \theta\frac{dS}{S} + \phi\frac{dB}{B} \tag{2.1}$$

for weighting functions θ and ϕ. By adopting this formalism, we may drop the explicit reference to A and use S.

A portfolio, generated by these weighting functions, is called self financing if $Sd\left(\frac{L}{S}\theta\right) + Ld\left(\frac{L}{B}\phi\right) = 0$. This condition means that the portfolio's return could be generated by continuously rebalancing (buying and selling at zero cost) between risky and risk-free assets. A sufficient condition for the liability portfolio, defined in this manner, to be self financing is that $\theta + \phi = 1$. Generally, the weighting functions ϕ and θ could be real valued mappings on the triple (t,S,L). However, we only consider the special but simple case, where the weighting functions are linear mappings in $\frac{1}{q}$ where $q = \frac{L}{S}$ is called the coverage ratio.

The future coverage ratio measures the performance of the liability with respect to the assets' performance on a path-by-path basis. Of course the inverse of the coverage ratio is the solvency ratio.

We will use: $\phi = a\frac{1}{q} + b; \theta = 1 - \phi$ where (a,b) represent two real valued parameters. Ultimately these parameters define the quantitative contribution of risky and risk-free returns, dependent on the solvency ratio, to the liability.

The process for the coverage ratio by Ito's Lemma is then

$$\frac{dq}{q} = (\theta - 1)((\mu - \sigma^2)dt + \sigma dZ) + \phi r dt. \tag{2.2}$$

The reserve is denoted by $R = A - L$ and it can be positive or negative. The balance sheet to the insurer can be simplistically represented, in Table 1, where the reserve has been split in its positive and negative parts:

Table 1 Balance sheet

ASSETS	LIABILITIES
A	L
R^+	R^-

3. FRAMEWORK

3.1 Smoothing returns

Although there is no standard notion of smoothing returns, it is usually present in a with-profits product. We start this section by discussing the effect of arithmetic return smoothing. Let us assume that the returns are declared annually. The value of the reference portfolio of assets is denoted by A_i at the end of year i. The return is then $r_i = \ln \frac{A_i}{A_{i-1}}$. The return of the liability in year i will be denoted by $s_i = \ln \frac{L_i}{L_{i-1}}$. The simplest mechanism to smooth is to average returns of the asset portfolio; for example, we use the 5-year average $s_i = \frac{1}{5} \Sigma_{j=0}^{4} r_{i-j}$. The liability will then evolve according to: $L_i = L_0 e^{\sum_{j=1}^{i-5} r_j + \frac{4}{5} r_{i-4} + \ldots + \frac{1}{5} r_i}$. The annualised variance of $\ln L_i$ is given by $\sigma^2 \left(1 - \frac{2}{i}\right)$ for $i > 5$, where σ is the annual return volatility of A. Over the minimum of 5 years the expected annual volatility of the liability will be $\sigma \sqrt{1 - \frac{2}{5}} \approx 0.78\sigma$, a reduction of 22%. Over 10 years the effect of smoothing returns will result in an expected volatility reduction of only 10% $(\sigma \sqrt{1 - \frac{2}{10}} \approx 0.9\sigma)$. Clearly this will make the short term volatility of the liability lower than that of the reference asset and less so for longer periods. So the impact of averaging is limited.

We later see that the liability's volatility is drastically reduced when combined with a guarantee and bonus framework. The introduction of a bonus appears in section 3.3. We do not consider return averaging explicitly for the rest of the paper since the effect is limited and the effect could be considered by adjusting the asset volatility.

The cumulative annual volatility of the liability is materially reduced by linking its instantaneous return to the risk-free rate with low volatility. We call this reserve smoothing since it can be equated to keeping the reserve in cash although that is not generally the case. Such a commercial mechanism is analysed in Hansen and Miltersen (2002) and Miltersen and Persson (2003). Their definition is discrete and based on the accounting concepts of the client and reserve account. Because we view various ideas, defined continuously, as a stochastic process we now define the liability using a SDE. Such a SDE represents the accrual rule defining the notional value for the liability (in terms of various other known variables). Our particular choice of accrual rule could be constrained by what may be intuitively acceptable.

3.2 Reserve Smoothing

We assume that all assets are held in risky assets, collectively represented by S. We let the liability participate directly in the assets' returns, and excess reserve is distributed (resp. shortfall is charged) as defined by a mean reverting process. For our purposes a process that pays out excess reserves and recoups for a shortfall over time would be a mean reverting process. In practice, the excess (or shortfall) reserve may be defined in a proportionate way relative to a target solvency ratio. A simple mathematical way to define a mean reverting process is by using an SDE of a special form.

Let us proceed by letting the liability participate in the risky returns by attributing a fraction of risky returns to the liability. Hence the liability will potentially accrue some of the higher expected risky returns. In addition, risky assets in excess of a liability multiple is transferred to the liability at a fixed rate. The liability multiple would be the liability multiplied by target solvency ratio. Conversely, a shortfall to the liability multiple is recouped by reducing the liability return. In other words, we consider the liability where it is defined by the SDE

$$dL = \rho dS + \alpha\left(S - \beta L\right)dt = \rho\frac{1}{q}\frac{dS}{S} + \alpha L\left(\frac{1}{q} - \beta\right)dt,$$

where $0<\rho\leq 1$ denotes the participation rate, $\beta\approx 1$ denotes the target solvency ratio, and α is the reserve distribution or charging rate. Since $R=S-L$ the reserve then satisfies

$$dR = \left(1-\rho\right)dS - \alpha\beta\left(R - \left(1-\frac{1}{\beta}\right)S\right)dt.$$

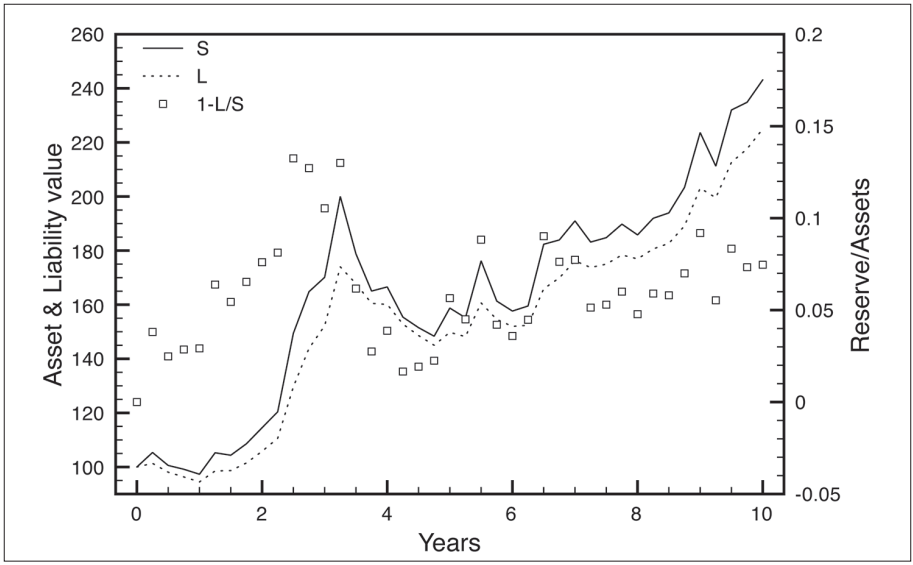

FIGURE 3.1

Clearly the reserve reverts to the level $\left(1-\frac{1}{\beta}\right)S$ relative to the assets, with reversion rate $\alpha\beta$ (if β close to 1 then $1-\frac{1}{\beta}\simeq\beta-1$). Figure 3.1 illustrates a sample path over 10 years (to construct the example we did not use the continuous time accrual but a discrete time version where the steps are quarterly) where μ=0.09, σ=0.15, ρ=0.6, α=1.5 and β=1.05 for the asset with the corresponding defined liability and reserve as percentage of assets.

Note that the instantaneous variance of the liability is then $(\rho\sigma)^2$ and this may still transfer substantial volatility to the liability. When ρ=0 the instantaneous volatility is 0 and the liability's only return comes from paying out excess reserves, as represented by the term $\alpha\left(\frac{1}{q}-\beta\right)dt$. We explore this case next to see what the effective volatility of the liability would be.

Writing the process for the liability in our standard format is

$\frac{dL}{L}=\frac{\alpha}{r}\left(\frac{1}{q}-\beta\right)rdt+\frac{\rho}{q}\frac{dS}{S}=\frac{\alpha}{r}\left(\frac{1}{q}-\beta\right)rdt$, if ρ=0. The solution to this SDE, after some algebraic manipulation, is

$$L_t=L_0+R_0+e^{-\beta t}\left(\alpha S_0-R_0\right)+\int_0^t\left(1-\alpha\left(1-e^{-\beta(t-u)}\right)\right)dS_u.$$

We interpret this formula next. In continuous time, the P&L of holding x units of S at time u would be xdS_u and an integral $\int_0^t xdS_u$ represents the cumulative P&L from trading S. The last integral term of the equation above is of such a form, where the holding in S at time u is $\left(1-\alpha(1-e^{-\beta(t-u)})\right)$. It means that the return of L_t can be seen as a weighted return of S_u; $0<u<t$. The volatility of L_t is determined by these weightings where the initial weighting in S is $\left(1-\alpha(1-e^{-\beta t})\right)$, which is below 100%. The weighting grows to 100% at $u=t$. This drives the smoothing of returns of S in L. We demonstrate the magnitude of the reduction for different target solvency ratios and reserve payout rates next.

L_t is an Ito Integral and its variance is given by the integral $\int_0^t\left(1-\alpha\left(1-e^{-\beta(t-u)}\right)\right)^2\sigma^2S_u^2du$, which is not tractable since S is stochastic. However, we may obtain an upper bound using Jenson's inequality. It follows that this integral is bounded above by

$$\left((1-\alpha)^2\sigma^2t+2(1-\alpha)\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)\sigma^2+\frac{\alpha^2\sigma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right)\int_0^t S_u^2du.$$

Also note that since $S_t=S_0+\int_0^t S_u\left(\mu du+\sigma dZ_u\right)$ the variance of S_t is given by

$\sigma^2 \int_0^t S_u^2 du$. Thus the ratio of variance of L_t to S_t is given by the leading term in brackets and it specifies the volatility reduction as a consequence of reserve smoothing. Figure 3.2 illustrates the upper bound of average annual volatility (the square root of the leading term in brackets divided by t) of L_t as a function of the payout rate α, for σ=0.15 and t=10.

It demonstrates how effective reserve smoothing could be since the standard deviation of L_{10} could be lower than 4% when the risky asset volatility is 15%. In due course, in section 4, we consider a wide range of strategies and we demonstrate that there are many strategies that lower the terminal variance of L compared to the variance of S.

3.3 Bonus formula

With-profits funds accrue a guaranteed return with a discretionary bonus, and we develop the logic for a simple vesting bonus over an accrual period. By retaining some of the asset returns in the reserve, a mechanism is required to transfer returns to policyholders. In the previous section, we defined the mechanism using an accrual rule premised on a mean reverting reserve level relative to the asset value.

With-profits returns incorporate two accrual components. One part accrues the guaranteed return. The second component is a vesting bonus based on various factors including: the excess (average) return, current reserves or lack thereof (equivalently the solvency ratio), returns from competing products and expected future market

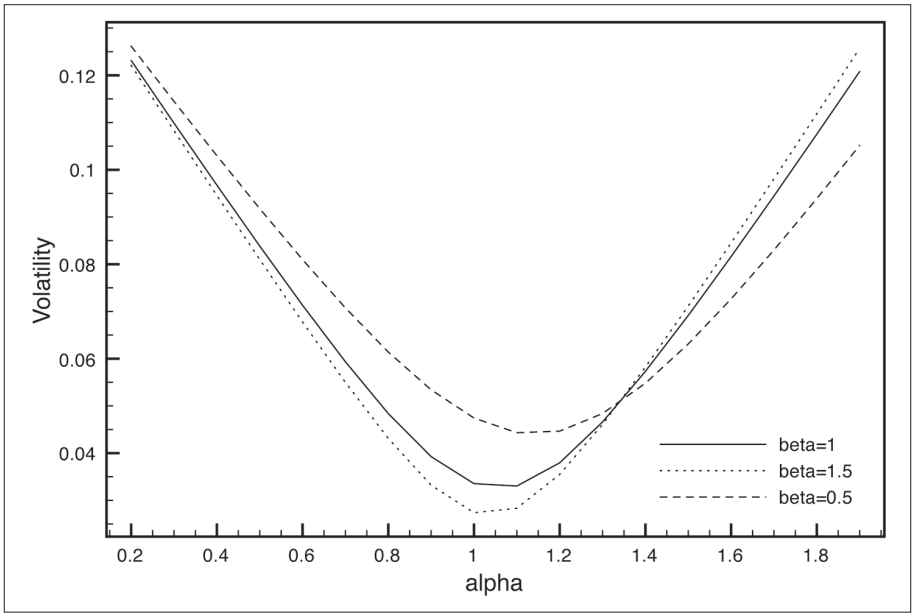

FIGURE 3.2

conditions. To the extent that the bonus is dependent on excess returns and the asset liability ratio, we present a formula that represents a vesting bonus.

Since we work discretely over an accrual period, we digress from the continuous time setting briefly (we introduce the discrete time setting in section 4.1). For the moment subscripts will identify the parameter for a particular interval indexed with i. Suppose, by contractual agreement, that there is a guaranteed rate of return g per year. The excess return is max $(0,r_{i+1}-g)=(r_{i+1}-g)^+$. If a fraction ρ of this is added to the return of the liability, then the coverage ratio would become (denoted by the hat) $\hat{q}_{i+1} = q_i e^{\rho(r_{i+1}-g)^+-(r_{i+1}-g)}$. This formula reflects that, if the risky returns are below the guaranteed rate the coverage ratio would decrease by $g-r_{i+1}$. On the other hand, if the risky return exceeds the guarantee rate then the coverage ratio will decrease by $(1-\rho)(r_{i+1}-g)>0$. In general, $\hat{q}_{i+1}$ is unlikely to be equal to the target coverage ratio. Suppose the fund targets a coverage ratio e^{β}. To create a situation where the coverage ratio moves closer to the target, the excess return is adjusted upwards or downwards based on the difference between $\hat{q}_{i+1}$ and the target solvency ratio, if at all possible. The adjustment is affected by adding a fraction γ of the difference: $\gamma\left(\beta - \ln \hat{q}_i\right)$. The vesting bonus (excess return and adjustments for coverage ratio) becomes

$$b_{i+1} = \left(\rho\left(r_{i+1}-g\right)^+ + \gamma\left(\beta - \ln q_i - \rho\left(r_{i+1}-g\right)^+ + \left(r_{i+1}-g\right)\right)\right)^+,$$

and liability will then be: $L_{i+1} = L_i e^{g+b_{i+1}}$. In summary

$$b_{i+1} = \begin{cases} \left(\left(\rho+\gamma\left(1-\rho\right)\right)\left(r_{i+1}-g\right)+\gamma\left(\beta-\ln q_i\right)\right)^+; & \text{if } r_{i+1} > g, \\ \gamma\left(\beta - \ln q_i + \left(r_{i+1}-g\right)\right)^+; \text{if} & r_{i+1} < g. \end{cases}$$

This bonus formula shows that the policyholder only has upside exposure to the risky return to the extent that the risky return exceeds the guarantee rate by the term $(\rho+\gamma(1-\rho))(r_{i+1}-g)$. The rest of the bonus, i.e. $\gamma(\beta-\ln q_i)$ reflects additional bonus to the extent that the remaining return $(r_{i+1}-g)$ exceeds the target coverage ratio deficit $(\ln q_i-\beta)$. If the coverage ratio is lagging the target then a portion is also taken to reserves.

We would like to compare the accrual mechanism to others and to this end we need to ignore the bonus mechanism (ignore the guarantee rate mechanism defined by the max and g parts). Then the liability return would be $(\rho+\gamma(1-\rho))r_{i+1}+\gamma(\ln q_i-\beta)$, and in continuous time we could write $\frac{dL}{L} = \left(\rho+\gamma\left(1-\rho\right)\right)\frac{dS}{S} + \gamma\left(\frac{S}{L}-\beta\right)dt$. This form is analogous to the SDE defining reserve smoothing. The non-guaranteed version of the return can be viewed as that of a portfolio where the asset weightings depend linearly on the solvency ratio.

3.4 Portfolio Insurance

Portfolio insurance is a concept that grew from dynamically hedging an option, thereby creating the option payoff, particularly before the Wall Street crash in 1987.

A simple put option dynamic hedging strategy accounts for portfolio insurance at a chosen point in time, which is a significant limitation from a practical point of view. Usually it is applied over a short time horizon which is rolled forward as time passes.

Other strategies that have been used for portfolio insurance include: Constant Proportion Portfolio Insurance (CPPI) (Balder, Brandl & Mahayni, 2009; Boulier & Kanniganti, 1995; Black & Jones, 1987), direct use of options and other derivatives on an ongoing basis found in Variable Annuities (Blamont & Sagoo, 2009), and more recently the definition of dynamic fund protection (Gerber & Pafumi, 2000), which has some similarity to a look-back option but on a contingent notional amount. As the protection becomes more effective the cost of the strategy also increases and it can be too expensive in practice.

For our purposes we define the liability in a way that the policyholder has participation to the upside (to some degree) and less risk on the downside (again to some degree), as opposed to using a CPPI or other portfolio insurance strategy. To this end we take the cue from the definition of CPPI, defined by the SDE

$$\frac{dA}{A} = m\left(1 - \frac{F}{A}\right)\frac{dS}{S} + \left(1 - m\left(1 - \frac{F}{A}\right)\right)\frac{dB}{B}.$$

This process protects the portfolio with value A from a floor level, denoted by F_t. The parameter m represents a leverage parameter. If the portfolio value falls to the floor value there would be no exposure to the risky asset and all investments would be in bonds that accrue to the guaranteed maturity value. Of course various additional features are introduced to CPPI from a commercial perspective. The behaviour of a CPPI strategy is well understood and studied. We adapt this approach so that the liabilities follow asset returns to the extent that the liability lags the assets, but accrue at the risk-free rate otherwise to limit the exposure to the insurer.

To this end we modify the strategy as follows. Risky returns accrue proportional to the amount that the liability is behind the assets. In other words the process for the liability will be

$$dL = m\left(S - \beta L\right)\frac{dS}{S} + \left(L - m\left(S - \beta L\right)\right)\frac{dB}{B}, \tag{3.1}$$

for some parameters $m>0$ and $\beta>0$. The parameter m is an exposure leverage parameter and β represents the relative amount that the assets should exceed the liability. Because the implied weightings add up to one, the portfolio is self financing and consequently we know the fair value of L_t to be the initial capital invested L_0. The distribution of L appears in Figure 3.3 where $m=2$, and $\beta=1.05$, and it indicates that the liability will perform better than the risky asset on the downside.

We obtained this figure by using the solution of (3.1). To this end choose $a=-m$ and $b=1+m\beta$, in the Appendix. For our strategy m and β are non-negative and therefore $1+m\beta\neq 0$. Subsequently we obtain

$$L_t = \frac{m}{1+m\beta} S_t + \left(\frac{L_0}{S_0} + \frac{-m}{1+m\beta} \right) S_0^{1+m\beta} S_t^{-m\beta} e^{(1+m\beta)\left(r+\frac{1}{2}\sigma^2\beta\right)t}.$$

It is straightforward to verify that, if $m>0$ and $\left(\frac{L_0}{S_0} + \frac{-m}{1+m\beta}\right) \geq 0$, then $L_t \geq 0$. This is relevant to decide on parameters but this will not be generally true when we move to discrete time. This formula indicates that the liability does not depend on the particular path that the risky assets follow, similar to CPPI. The formula also explains that the liability is the same as a long or short position in the risky asset:

— the leading term shows a long position in S_t to the amount $\frac{m}{1+m\beta}$,

— assuming that $\left(\frac{L_0}{S_0} + \frac{-m}{1+m\beta}\right) \geq 0$, the second term shows a non linear short exposure in S_t, that becomes bigger in size as S_t becomes smaller, because of the negative power $-m\beta<0$,

— the notional size of this total exposure becomes shorter and bigger in size as S_t is smaller, dominated by the second term.

If the risky asset underperforms, then the non-linear short exposure in the second term creates some downside out-performance for the liability.

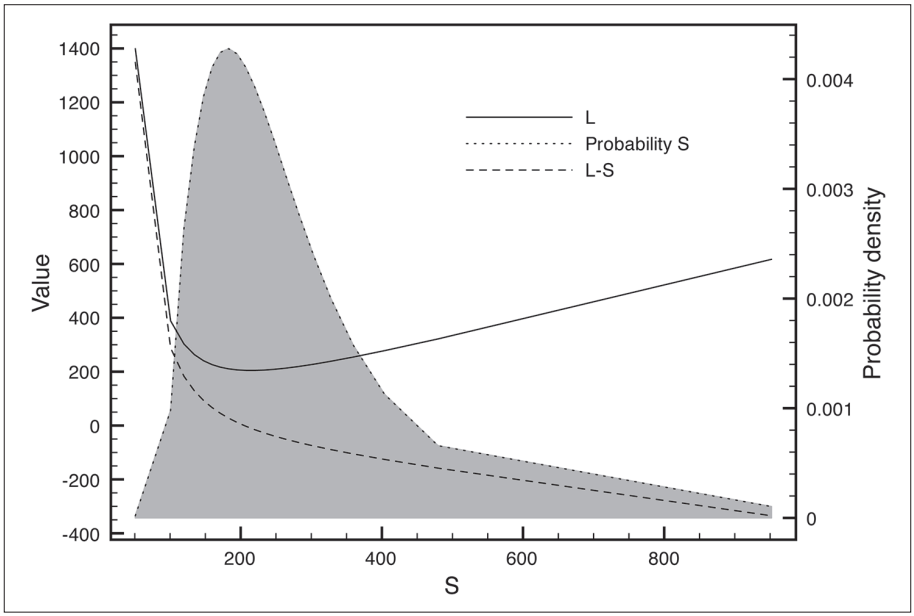

FIGURE 3.3

3.5 Derivatives and Conceptual Considerations

Derivatives' application to create an investment with downside protection and upside participation are numerous. The difference to the approaches introduced above is that derivatives are defined with a payoff as opposed to an accrual rule. A derivative payoff depends on variables: the asset values at selected times. Examples of such products are Variable Annuities (Blamont & Sagoo, 2009). There is a great deal of similarity with the above concepts however. For example, bonuses that vest would be comparable to a ratcheting strike option – called a Cliquet option. Other option payoffs that are used would include Look-back options and Average-rate options.

In fact, in recent times the trend has been to view the guarantee exposure as a complex and exotic derivative. However, because an insurance product is generally not easily defined by a payoff at a point in time, derivative models for the value of the guarantee are usually implemented as Monte Carlo simulations of the referenced asset values or prices. These valuation, and consequent hedging, models require substantial investment in computational infrastructure as well as ultimately accepting some numerical error due to the slower rates of convergence from the Monte Carlo simulation of long dated and heavily path dependent payoffs. It is also rare that the market value of the asset is computed and reported in a like manner since that belongs to the policyholder. Shareholders bear the burden of the minimum guarantee in the end although some risk sharing may apply.

It is also interesting to ask the question whether it is possible to follow a dynamic asset allocation in such a way that the resulting investment has a guaranteed minimum rate of return with equity upside. This is generally not possible as explored in Kleinow and Willder (2007) and Kleinow (2009).

We indicate structural similarities between smoothing, reserve distribution and portfolio insurance through their weighting functions, in Table 2 below. All these simplistic definitions of a liability accrual rule have portfolio weighting terms that depend linearly on the solvency ratio and include a fixed return component. It is the fixed return component that induces a smoothing of returns of the associated defined liability. When making well-considered choices of parameters, for example as with a bonus formula, the resulting liability may not be self financing and the mark to market of the asset is different to the book value. By using self-financing affine strategies to define the liability we always have the mark to market of the liability equal to the book value and the accrual rule fully defines the liability.

3.6 Summary

We summarise our conceptual SDE framework in Tables 2 and 3. Table 2 gives a descriptive comparison and Table 3 an algebraic comparison. These tables makes it simple to compare the accrual mechanism defining the liability.

Reserve smoothing differs from the *Bonus* in the attribution of risky returns, i.e. in the θ component. The structure for the bond return is equivalent. We explore the characteristics of the parametric case, defined in discrete time, in the next section. The

parametric case represents all simple mechanisms of dynamically re-balancing a self-financing portfolio, driven by the solvency ratio.

Table 2 Comparison of accrual logic for different liability definitions

Scheme	Bonus		Reserve Smoothing		CPLI	
Mechanism	Risky	Bond	Risky	Bond	Risky	Bond
Fraction (based of liability)	X					
Fraction re-based to asset level			X			
Proportional to excess solvency		X		X	X	
The balance/self financing						X

Table 3 Weighting function comparison

Scheme	θ	ϕ
Bonus	ρ	$\gamma\frac{1}{q}-\gamma\beta$
Reserve smoothing	$p\frac{1}{q}$	$\alpha\frac{1}{q}-\alpha\beta$
CPLI	$m\frac{1}{q}-m\beta$	$-m\frac{1}{q}+(1+m\beta)$
Parametric	$-a\frac{1}{q}+(1-b)$	$a\frac{1}{q}+b$

4. DISCRETE TIME

4.1 Definitions

There are various options to define the liability in discrete time in the spirit of the continuous time approach. It is also essential to move to discrete time to make it possible to define the bonus and guarantee mechanism properly. We want to fix the weightings over an interval. In general we denote the fixed weightings θ_i and ϕ_i using information up to the end of period i. From continuous time (using Ito's Lemma) we have

$$d\ln L = \left(\phi_i r + \frac{1}{2}\theta_i\left(1-\theta_i\right)\sigma^2\right)dt + \theta_i d\ln S.$$

If the weightings are fixed over a time interval, then the solution to preceding SDE is

$$L_{i+1} = L_i e^{\theta_i r_{i+1} + \left(\phi_i r + \frac{1}{2}\theta_i(1-\theta_i)\sigma^2\right)\delta},$$

where $r_{i+1} = \ln\left(\frac{s_{i+1}}{s_i}\right)$. This formula motivates a possible definition of the liability in discrete time. However, to replicate this liability in reality would require continuous rebalancing and, because we wish to fix the implicit asset weightings in an interval, we do not use this definition. We rather adopt the following definition:

$$L_{i+1} = L_i + L_i\theta_i\left(e^{r_{i+1}} - 1\right) + L_i\phi_i\left(e^{r\delta} - 1\right),$$

when there is no guarantee. This is well defined regardless of the choice of weightings θ_i, ϕ_i. When the liability is self financing, i.e. $\theta_i + \phi_i = 1$, we get

$$L_{i+1} = L_i + L_i\theta_i e^{r_{i+1}} + L_i\phi_i e^{r\delta} = L_i\left(1 + \phi_i\left(e^{r\delta} - e^{r_{i+1}}\right)\right). \tag{4.1}$$

To cater for a guarantee we modify the liability accrual rule as follows. G^ρ denotes a guaranteed liability, where the parameter ρ represents the participation ratio, as it is defined

$$G^\rho_{i+1} = G^\rho_i e^{g\delta} + G^\rho_i \max\left[0, \rho\left(\theta_i e^{r_{i+1}} + \phi_i e^{r\delta}\right) - e^{g\delta}\right]. \tag{4.2}$$

The participation ratio is required so that we may price the guarantee and it represents a reduction in effective notional exposure. For the liability L we have a self-financing portfolio strategy, consequently we have the fair value (also called risk-neutral value) of the liability is its initial value, i.e. the capital initially invested. In mathematical finance, see for example Baxter and Rennie (1996), we write this as $\mathrm{E}^Q\left(\frac{L_T}{B_T}\right) = L_0$, where the expectation E^Q refers to the risk neutral expectation (for simple Geometric Brownian Motion this expectation is computed by setting $\mu = r$). However, for a guaranteed liability it is typical that its fair value exceeds the initial investment when the participation level is 100%. That would translate into a loss to the insurer which is not fair. In the analysis that follows we found the participation ratio $\hat{\rho}$ (with respect to selected parameters) such that the liability's fair value is equal to the initial investment, i.e. that $\mathrm{E}^Q\left(\frac{G^{\hat{\rho}}_T}{B_T}\right) = G_0$. We therefore suppress dependency on the participation ratio in the notation below.

We consider weightings in the parametric form that will be linear in $\frac{1}{q}$:

$$\phi_i = a\frac{1}{q} + b\,;\ \theta_i = 1 - \phi_i = -a\frac{1}{q} + (1 - b).$$

For practical reasons it will be useful to add constraints on the value of the bond weighting ϕ_i : bounded from above by u and from below by l:

$$\tilde{\phi}_i = \max\left[l, \min\left[u, \phi_i\right]\right],\ \tilde{\theta}_i = 1 - \tilde{\phi}_i.$$

Unless stated we hence use the constrained parametric weightings with u=1 and l=0, which exclude any leverage.

4.2 Analysis

We take a moment to position what comes next. The parameters (a,b) define the way risky returns, bond returns, and reserves are distributed to the liability L and guarantee liability G. They cover mechanisms from fixed asset weightings to fast changing weightings depending on the solvency ratios along a path of returns. At some fixed maturity, representing a payout date, the assets could be above or below the respective liability values. This creates exposure to the insurer since accounting reserves are assumed to be invested in risky assets. Policyholders would prefer that the liability value is higher if the market ends up "lower" than a reference point, say the initial value accrued at the inflation rate. These concepts are illustrated in Figure 4.1 which shows one sample path. In the rest of the paper we describe what the impact of parameters are on expectations of the quantities introduced in the graph.

The rest of this section is arranged as follows. We first take a detailed look at the impact of the parameters on the relative performance of the liability. This will create a general understanding of the parameter impact. Then we analyse the performance, risks and costs of the liabilities from an economic perspective. This clearly identifies which parameter choices are of interest. We also illustrate the weighting functions to these parameters which is relevant from a practical point of view.

We wish to describe how the liability L and guarantee G behaves outright, and relative to S, particularly as a consequence of choice of parameters (a,b). The performance of the liability relative to the asset is considered by computing the

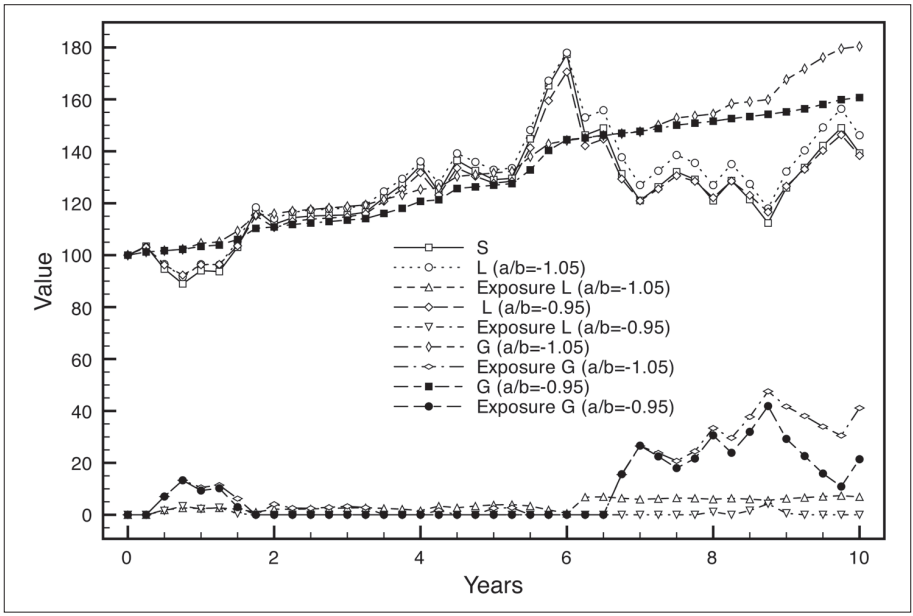

FIGURE 4.1

expectations of L conditional on the terminal value of S: $\mathrm{E}(L_T|S_T\in[x,y])$. We choose to use 5 equi-probable intervals for the terminal state of S_T denoted by I_i for $i=1,2,3,4,5$:

$$I_i = \left\{ S_T = S_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z} \mid Z \in [F^{-1}\left(\frac{(i-1)}{5}\right), F^{-1}\left(\frac{i}{5}\right)\right\},$$

where F^{-1} denotes the inverse cumulative probability density function of the Normal distribution. We denote the conditional expectations, using non-italic Roman characters: $\mathrm{q}-i=\mathrm{E}\left(\frac{L_T}{S_T}|S_T \in I_i\right)$ for $i=1,2,3,4,5$.

Throughout the analysis we used the following parameter values: Starting asset and liability values = 100. The time horizon is 10 years with quarterly rebalancing. The expected asset return is 9% and the risk-free rate is 6%. The guaranteed liability has a guarantee rate of 2% and bonuses are vesting quarterly. The asset returns are subject to an annual volatility of 15%. In summary: $S_0=L_0=100$, $T=10$, $\delta=0.25$, $\mu=0.09$, $r=0.06$, $\sigma=0.15$, and $g=0.02$.

We digress for a moment and consider the continuous case where the cover ratio q is described by the following SDE:

$$dq=q\theta((r-\mu+\sigma^2)dt+\sigma dZ),$$

with solution (see Appendix)

$$q_t = \left(q_0 + \frac{a}{b}\right) e^{-b\left(\mu-r-\sigma^2\left(1-\frac{1}{b}\right)\right)t-b\sigma Z_t} - \frac{a}{b}.$$

Its expectation is

$$\mathrm{q}_t = \mathrm{E}\left(q_t\right) = \left(q_0 + \frac{a}{b}\right) e^{-b\left(\mu-r+\sigma^2\right)t} - \frac{a}{b}.$$

This expectation represents the expected performance of the liability with respect to the risky assets. Figure 4.2 illustrates the expected coverage ratio at 10 years as function of the parameters (a,b). Recall that these parameters affect the contribution and the impact of the solvency ratio of risky and risk-free return to the liability's return. It is easier to understand the impact of changes of these parameters by depicting the expected coverage ratio as it changes for different values of the ratio $\frac{a}{b}$, and for different values of $a+b$. We use the ratio and sum through the rest of our analysis of L and G.

We provide the same illustration for the conditional expectations $\mathrm{q}-i$. The calculation of these values was by Monte Carlo simulation for two cases: $a+b=0.2$ and $a+b=0.8$. We used a million sample paths, each with 40 quarterly steps (representing a 10-year holding period), and returns of the risky asset using pseudo random numbers. The calculation was implemented in Matlab using standard functions.

Figures 4.3 and 4.4 observe different behaviour of L with respect to S primarily by different choices of the ratio $\frac{a}{b}$. Also note that it is possible to realise negative values

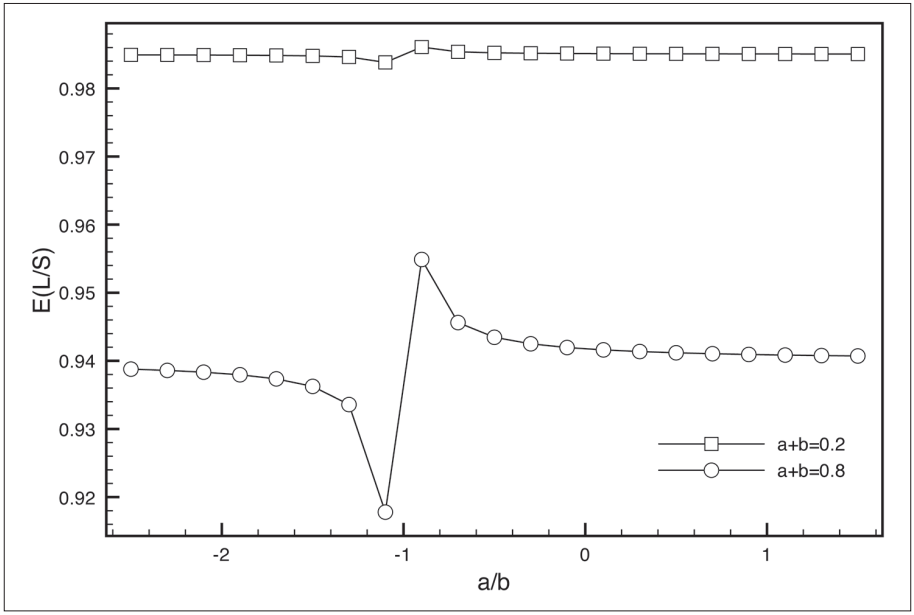

FIGURE 4.2

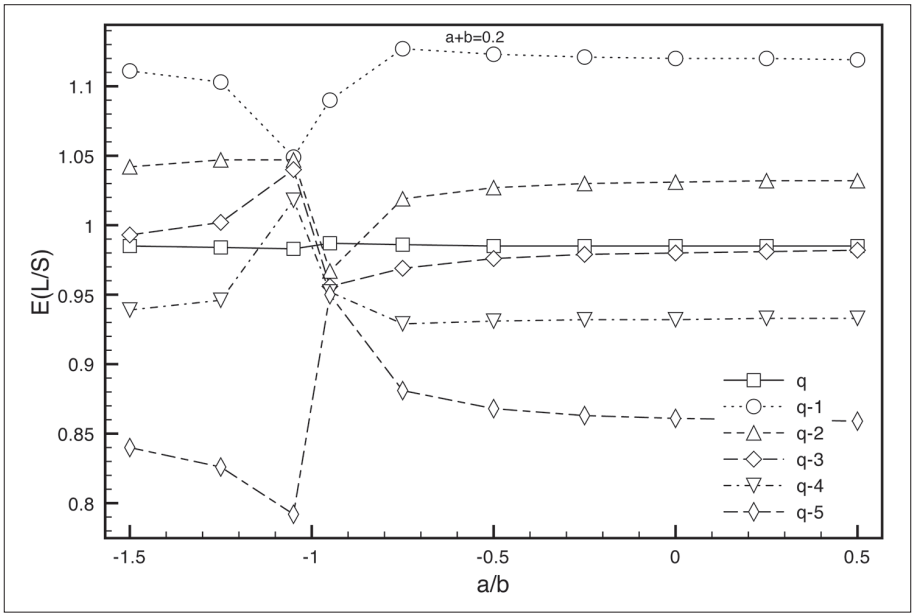

FIGURE 4.3

for the liability from Figure 4.4, since q assume negative values in some cases. The remedy is to constrain the weightings and for obvious reasons we chose bounds 0 and 1. For our purposes it is equivalent to assume that the effective portfolio representing the liability returns cannot be leveraged and no short selling of the assets is allowed. Systematically higher values for q–1 means that if assets end up to the downside, then the liability performed better than the risky asset. The conditional out-performance ranges widely and the interesting parameters are especially for ratios $\frac{a}{b}$ around –1.

We will later observe that intuitively acceptable strategies, i.e. with increasing exposure to risky assets when solvency ratios are higher, correspond to $\frac{a}{b} > -1$. The sensitivity of the return contribution, as the solvency ratio changes, increases as the ratio $\frac{a}{b}$ gets closer to –1. Also, note that if $a+b \neq 0$ then it is not possible that $\frac{a}{b} = -1$. The impact of $a+b$ can be thought of as the bond weighting when the solvency ratio equals 1.

The conditional expectations indicate parameter selection with downside out performance and upside participation. In reality the liability will result from one asset price evolution over time and so it is relevant to consider the variance of the q-i's. Figure 4.5 indicates the variances, associated with Figure 4.3 (i.e., for $a+b=0.2$). We

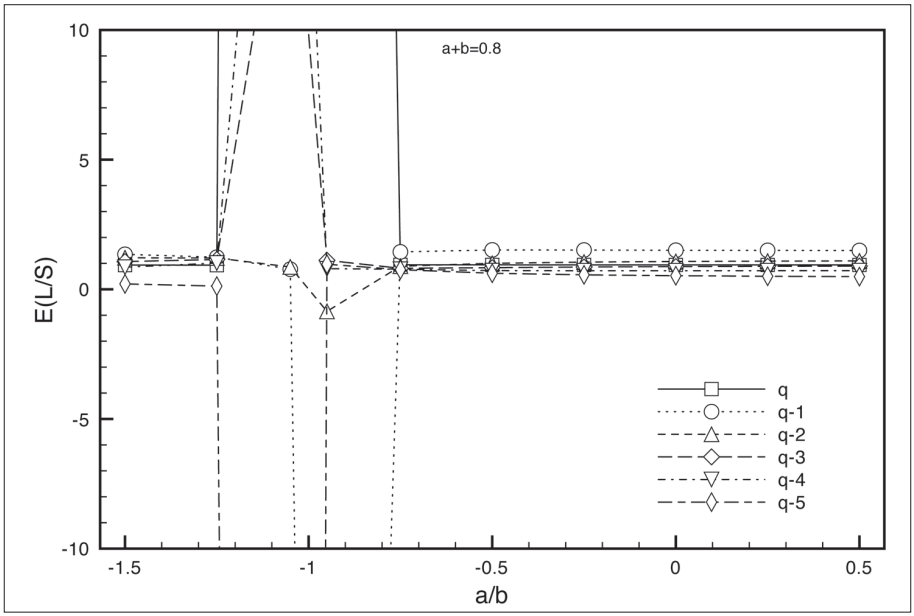

Figure 4.4

plot the standard deviations (note the logarithmic scale) for the conditional coverage ratio $\mathrm{Var}\left(\frac{L_T}{S_T}|S_T \in I_i\right)$ for $i=1,2,3,4,5$.

In conclusion we see significant expected out-performance on the downside for parameters choice where the ratio $\frac{a}{b}$ is around –1. The variance is substantially different on either side of –1. This means a policyholder could still realise a substantially different outcome and therefore it is useful to consider the guaranteed liability *G*. A guaranteed liability would have a direct cost to the insurer for the potential that the assets fall below the liability but also a different direct exposure to asset returns and this will reduce the returns of the guarantee liability. The above information is useful when chosing parameters (*a*,*b*) for an outcome that is desirable to a policyholder or shareholder and more metrics are required to this end. A policyholder in a with-profits policy would like market participation to the upside, i.e.. q–3/4/5 as high as possible. To cushion the market downside the policyholder seeks q–1 and q–2 to be as large as possible. Clearly there is a trade-off between these two goals and some preferences to the upside and protection are required to pick parameters.

The insurer is exposed to the extent that the assets fall below the liability, since the reserves are also invested in the assets. The market cost of this exposure is given for *L* by

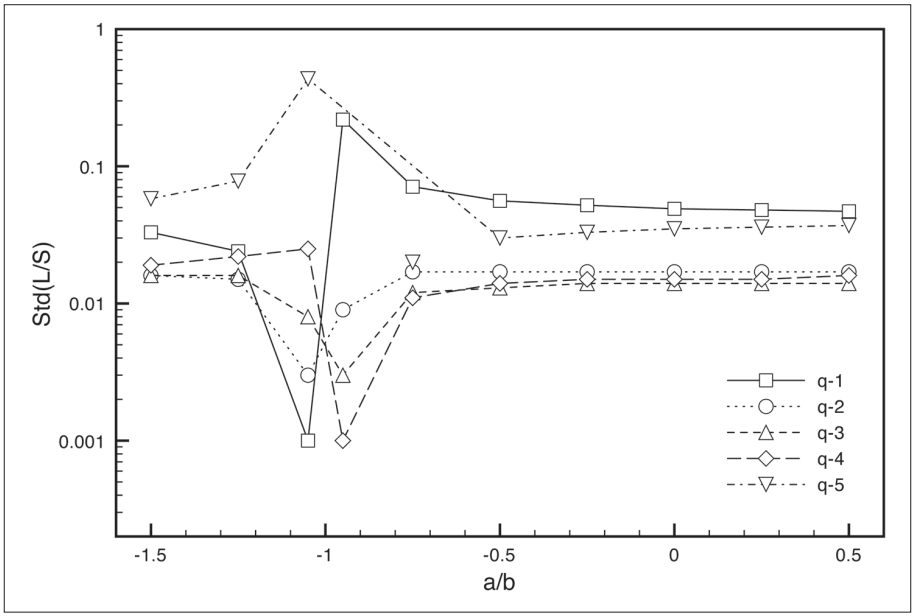

FIGURE 4.5

$$\mathrm{c} = \mathrm{E}^{Q}\left(\frac{\max\left(0, L_T - S_T\right)}{B_T}\right),$$

where the expectation is taken in the risk neutral measure. This cost is called the Investment Guarantee Reserve (IGR) and Figure 4.6 shows it with respect to different ratios $\frac{a}{b}$ (we always used an initial coverage ratio of 1, i.e. q_0=1). The figure indicates that the lowest IGR results when $a+b$ is smaller and then the ratio has limited impact. For other combinations there is no systematic impact. Note that the IGR for a guaranteed liability can be lower than for a non-guaranteed liability for $\frac{a}{b}$ around –0.5. Of course the expected return of the guaranteed liability is probably lower than in the non-guaranteed case.

To simplify the decision-making we do not consider the 5 conditional expectations. We pick a level of performance which will define the upside vs downside. In this context we choose this point to reflect a watermark return of 5% by using a level denoted $W_T=S_0e^{0.05T}=164.9$ for T=10. Then we define the downside/upside expectations, for example for S, as $S^{d/u}=\mathrm{E}(S_T|S_T</>W_T)$.

We now have all the components to indicate with respect to a wide selection of parameters (a,b) what the performance and risks would be. Tables 4 and 5 give

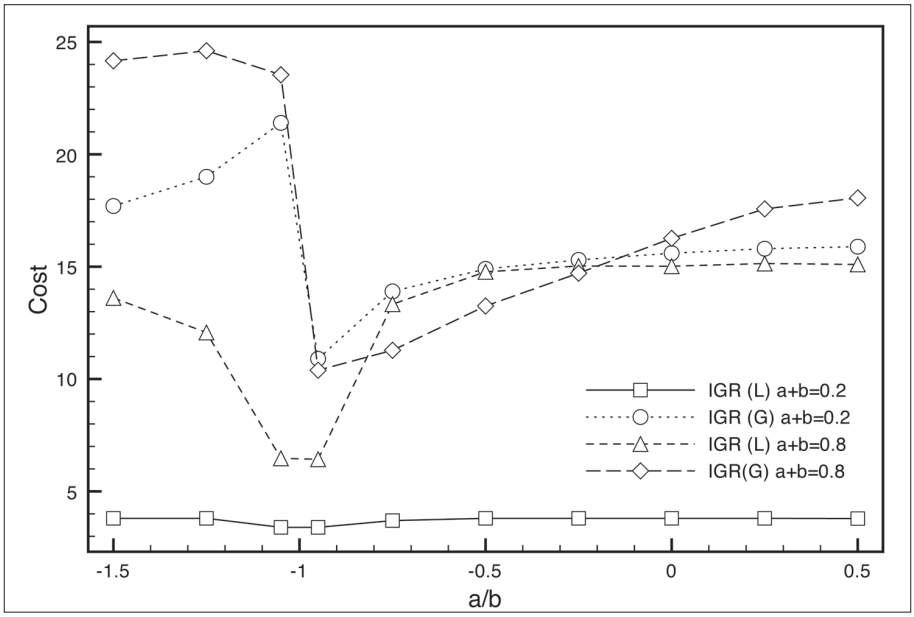

FIGURE 4.6

comprehensive raw results for the sake of completeness. It is not simple to identify useful behaviour from these tables and we investigate a more relevant "subset" later.

Table 4 Statistics for L_{10}

$\frac{a}{b}$	−1.5	−1.5	−1.05	−1.05	−0.95	−0.95	−0.5	−0.5	0	0	0.5	0.5
$a+b$	0.2	0.8	0.2	0.8	0.2	0.8	0.2	0.8	0.2	0.8	0.2	0.8
$E(L_T)$	230.6	231.9	228.4	197	237.2	203.4	231.6	233.1	231.6	199.3	231.7	193.5
$E(L_T\|S_T<164.9)$	137.9	138.0	132.7	164.9	131.9	155.1	138.0	136.2	138.0	168.9	138.1	171.8
$E(L_T\|S_T>164.9)$	265.3	266.9	264.3	209.0	276.9	221.4	266.4	269.1	266.4	210.7	266.9	201.6
$E^Q\max(0,L_T-S_T)$	3.8	3.8	3.4	13.6	3.4	12.1	3.8	6.4	3.8	14.8	3.8	15.0
$\mathrm{Std}(L)$	84.9	91.3	79.0	29.7	113.5	44.9	92.7	108.9	91.0	43.2	91.1	18.5
$\mathrm{Std}(L_T\|S_T<164.9)$	25.1	23.7	27.7	28.2	16.3	31.1	23.1	30.8	23.6	3.2	23.8	7.9
$\mathrm{Std}(L_T\|S_T>164.9)$	72.4	82.0	59.8	19.5	109.0	34.8	84.4	105.4	81.7	45.7	81.6	14.3

Table 5 Statistics for G_{10}

$\frac{a}{b}$	−1.5	−1.5	−1.05	−1.05	−0.95	−0.95	−0.5	−0.5	0	0	0.5	0.5
$a+b$	0.2	0.8	0.2	0.8	0.2	0.8	0.2	0.8	0.2	0.8	0.2	0.8
$E(G_T)$	183.2	191.3	176.4	172.7	206.4	173.0	194.2	210.4	191.8	203.1	190.8	189.2
$E^Q(G_T)$	99.6	99.9	99.9	100.1	99.7	100.3	99.8	99.7	99.9	100.0	99.9	99.8
$E(G_T\|S_T<164.9)$	179.3	173.5	188.5	196.3	156.2	195.4	170.8	153.2	173.0	164.0	173.7	175.2
$E(G_T\|S_T>164.9)$	184.7	197.9	171.9	163.9	225.4	164.6	202.8	231.6	198.8	217.8	197.3	194.5
$E^Q(\max(G_T-S_T,0))$	17.7	15.8	21.4	24.2	10.9	24.6	14.9	10.4	15.6	13.2	15.9	16.3
$\mathrm{Std}(G_T)$	13.7	18.1	30.2	35.3	46.1	39.7	23.6	56.3	19.0	47.3	17.5	12.8
$\mathrm{Std}(G_T\|S_T<164.9)$	12.6	10.2	19.3	34.5	17.7	33.4	10.2	17.5	10.2	8.5	10.3	6.4
$\mathrm{Std}(G_T\|S_T>164.9)$	13.8	15.7	32.2	31.3	38.7	38.6	21.2	50.6	16.6	47.6	15.2	10.4
ρ	40%	61%	44%	49%	46%	53%	40%	80%	40%	86%	40%	85%

Figure 4.7 shows the size of expected value, upside and downside performances, and costs. The figure illustrates the impact of dynamic asset allocation, driven by the solvency ratio, since parameters control the contribution of risky and risk-free returns of a self-financing portfolio. The figure also indicates the improvement of downside risk and performance by adding a guarantee mechanism. This comes with a reduction of performance on the upside.

To simplify and consolidate the raw results we compute annual returns, risks

and costs (from Tables 4 and 5) in Table 6. It provides a summary of the economic impact of different accrual rules.

Table 6 Return, Std and IGR for liabilities for $a+b=0.2/0.8$

a/b	L return	G return	Std L	Std G	Igr L	Igr G
−1.5	8.35%/6.78%	6.06%/5.47%	11.64%/4.76%	2.37%/6.46%	.37%/1.28%	1.64%/2.16%
−1.25	8.31%/7.1%	5.95%/5.48%	11.1%/6.98%	3.17%/7.26%	.37%/1.14%	1.74%/2.19%
−1.05	8.26%/8.15%	5.68%/5.54%	10.94%/13.96%	5.41%/7.16%	.34%/.63%	1.94%/2.11%
−0.95	8.64%/8.46%	7.25%/7.44%	15.13%/14.77%	7.06%/8.46%	.33%/.62%	1.04%/.99%
−0.75	8.47%/7.44%	6.82%/7.47%	13.23%/11.26%	4.64%/9.52%	.37%/1.25%	1.3%/1.07%
−0.5	8.4%/6.9%	6.64%/7.09%	12.66%/6.86%	3.85%/7.37%	.37%/1.38%	1.4%/1.24%
−0.25	8.43%/6.7%	6.56%/6.69%	12.59%/4.37%	3.4%/4.44%	.37%/1.4%	1.43%/1.38%
0	8.4%/6.6%	6.51%/6.38%	12.42%/3.02%	3.13%/2.13%	.37%/1.4%	1.45%/1.51%
0.25	8.41%/6.54%	6.49%/6.22%	12.45%/2.51%	2.99%/1.29%	.37%/1.41%	1.47%/1.62%
0.5	8.4%/6.52%	6.46%/6.12%	12.43%/2.36%	2.9%/1.25%	.37%/1.41%	1.48%/1.66%

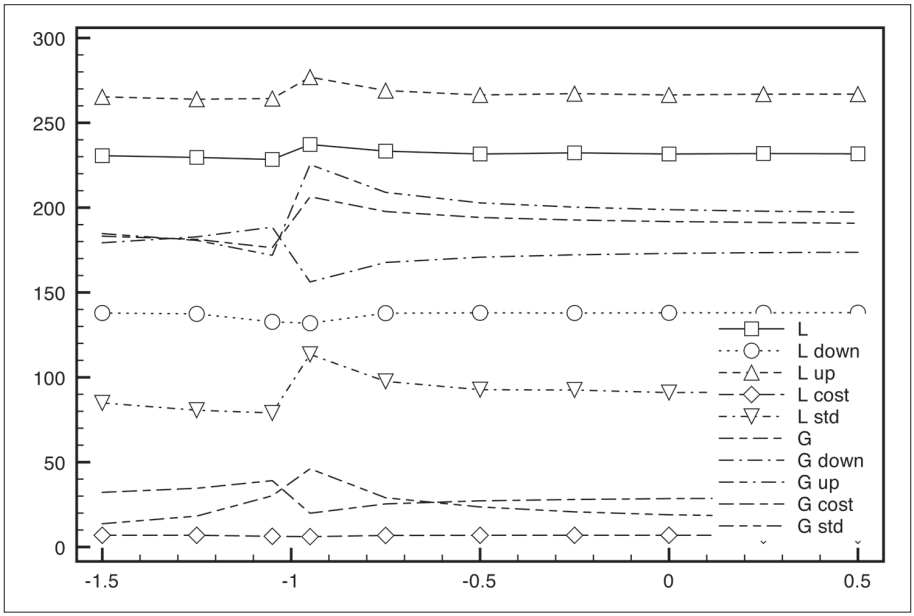

Figure 4.7

We now wish to focus on a subset of parameters, those where the ratio $\frac{a}{b}$ lies between –1 and 0, and take further steps to consolidate the results in terms of performance ratios.

We identify whether some of the considered liabilities are better suited from a with-profits perspective. Consider the weighting functions, illustrated in Figure 4.8, that result for the indicated choices of parameters a and b.

We see that, for $\frac{a}{b}$ between –1 and 0, we get weighting functions that are intuitive, i.e. the liability return weights the asset return higher for higher solvency ratios. For this subset, corresponding to intuitive with-profits accrual rules, we generated more granular results. Table 7 indicates the resulting returns, volatilities and Sharpe ratios with respect to a wide range of intuitive parameters.

These numbers show that some accrual rules may result in negative Sharpe ratios which one may wish to avoid. They also demonstrate an earlier claim that the volatility of a guaranteed liability is substantially lower. We summarise the high level impact of parameter choices next. Table 8 states the parametric choices that give the optimal choices in respect of returns, economic performance, risk to the policyholder and risk to the insurer.

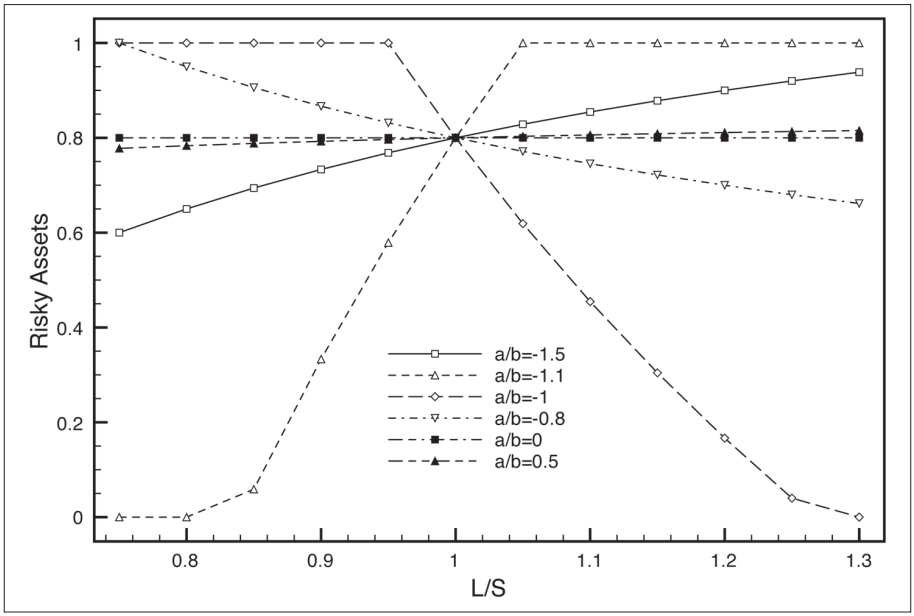

Figure 4.8

Table 7 Return, Std, and Sharpe ratio for L/G with respect to intuitive weightings

a+b / a/b	L/G	Std L/G	Sharpe ratio L/G
asset	9,0%	15,9%	0,189
0.20;–0.95	8.32%/6.22%	15.1%/7.1%	15.3%/3.1%
0.20;–0.70	8.07%/5.45%	13%/4.5%	15.9%/–12.3%
0.20;–0.50	8.05%/5.25%	12.7%/3.9%	16.1%/–19.4%
0.20;–0.30	8.03%/5.14%	12.6%/3.5%	16.2%/–24.8%
0.20;–0.10	8.02%/5.07%	12.5%/3.2%	16.1%/–28.8%
0.40;–0.95	8.11%/6.38%	15.2%/8%	13.9%/4.7%
0.40;–0.70	7.24%/5.88%	11.3%/6.3%	10.9%/–1.9%
0.40;–0.50	7.15%/5.51%	10.2%/4.9%	11.3%/–10.1%
0.40;–0.30	7.11%/5.26%	9.6%/3.9%	11.5%/–19.1%
0.40;–0.10	7.09%/5.11%	9.3%/3.2%	11.7%/–27.5%
0.60;–0.95	7.97%/6.43%	15%/8.3%	13.1%/5.2%
0.60;–0.70	6.51%/6.26%	10.5%/8.3%	4.8%/3.2%
0.60;–0.50	6.3%/5.8%	8.2%/6.3%	3.7%/–3.2%
0.60;–0.30	6.21%/5.38%	7%/4.4%	3%/–14.1%
0.60;–0.10	5.23%/5.04%	3.4%/2.9%	–22.2%/–32.6%
0.80;–0.95	7.83%/6.44%	14.9%/8.5%	12.3%/5.2%
0.80;–0.70	6%/6.35%	10.3%/9.4%	0%/3.7%
0.80;–0.50	5.52%/5.85%	6.9%/7.4%	–6.9%/–2.1%
0.80;–0.30	5.33%/5.44%	4.7%/5%	–14%/–11.2%
0.80;–0.10	5.24%/5.04%	3.4%/2.9%	–22.2%/–32.6%

Table 8 Summary of parameter effects on liability performance

Goal	L	G
Highest return	$a+b$ & $\frac{a}{b}$ smaller	$a+b$ larger & $\frac{a}{b}$ smaller
Highest Sharpe ratio	$a+b$ smaller & $\frac{a}{b}$ larger	$a+b$ larger than 0.2 & $\frac{a}{b}$ close to –0.95
Best downside protection	$a+b$ & $\frac{a}{b}$ larger	$a+b$ larger & $\frac{a}{b}$ smaller
Lowest risk	$\frac{a}{b}$ smaller	$a+b$ & $\frac{a}{b}$ larger
Lowest IGR	$a+b$ smaller & $\frac{a}{b}$ close to –0.95	$a+b$ larger & $\frac{a}{b}$ close to –0.95

A more detailed impact of the ratio $\frac{a}{b}$ is shown in Figure 4.9. for $a+b=0.5$. It indicates the effect of increasing the return sensitivity to the solvency ratio, i.e. when the ratio $\frac{a}{b}$ gets closer to –1, of attributing returns thereby creating efficiency in the liability.

5. CONCLUSIONS

In this paper we explored a special class of liability accrual rules where the asset allocation depends only on the coverage ratio. In continuous time, where the weightings are linear, the liability's evolution can be stated explicitly. We showed, for a specific set of assumptions about returns and volatility, what the impact of parameters would be in this class, and identified in particular those of interest to a with-profits savings product. We also considered the comparable guaranteed liability.

Given our framework and assumptions we apply the results of the analysis. For the highest Sharpe ratio one may choose a more equity weighted portfolio when $q=1$, e.g. select $a+b=0.2$. In order to retain some downside out-performance, it would be in order to choose $\frac{a}{b}$ further away from –0.95, e.g. $\frac{a}{b}=-0.75$.

Thus we get $a=-0.8$ and $b=1$. If a guarantee structure is appropriate for the risk preferences of a policyholder, one may identify which liability rule has an expected return that exceeds the risk-free rate, e.g., $a+b=0.5$ and $\frac{a}{b}=-0.95$, i.e., $a=-9.5$ and

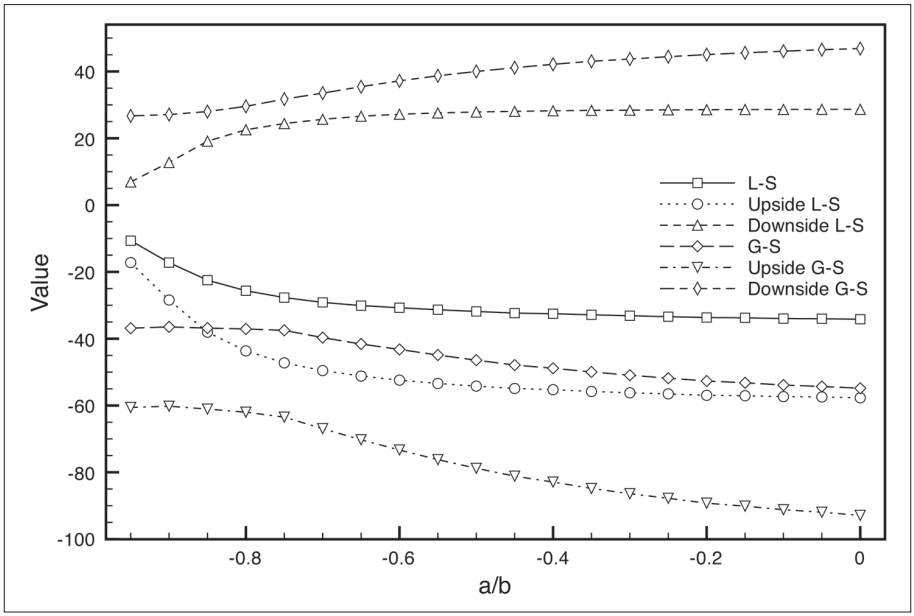

FIGURE 4.9

b=10. These parameters result in a lower IGR and meaningful participation giving a positive Sharpe ratio.

Our analysis above also indicates the following, which could be seen as obvious: the more the liability is defined to be like the risky asset, the lower the IGR and better the expected performance but with less downside out-performance; the performance of our liability with guarantee, when priced to par, materially reduced the participation and thus its expected return.

We emphasize that these conclusions are for our framework and assumptions. In practice other product factors would have an impact on performance and costs and it would be naive to extrapolate the above conclusion as good heuristics in any way. The analysis demonstrates how sensitive the performance could be for parameter choices and structural constraints. Further work, to draw stronger and more general conclusions, should include:

— the impact of assumptions (returns, volatilities and guarantee levels),
— possible changes due to return processes more reflective of real markets, and
— extending the product setting to allow for additional premiums and benefits to the policyholder.

In the end, the quest is to identify a combination of liability definition and product features that are demonstrably economically effective for the policyholder and insurer.

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APPENDIX

Solution to $\frac{dL}{L} = \theta\frac{dS}{S} + \phi r dt$ when $q(\theta-1)=-q\phi=-a-bq$

We shall find a solution for L by making use of the ratio of liability to assets: $q = \frac{L}{S}$. Application of Ito's lemma yields

$$dq = q\left(\theta - 1\right)\left\{\left(\mu - r - \sigma^2\right)dt + \sigma dZ\right\} = \left(-a - bq\right)\left\{\left(\mu - r - \sigma^2\right)dt + \sigma dZ\right\}$$

If $F = \ln\left(-\hat{a} - \hat{b}q\right)$ it follows that

$$dF = -b\left(\left(\mu - r - \sigma^2\left(1 - \frac{b}{2}\right)\right)dt + \sigma dZ\right),$$

which can be integrated explicitly. Thus, if $b \neq 0$, then

$$q_t = \left(q_0 + \frac{a}{b}\right)e^{-b\left(\left(\mu - r - \sigma^2\left(1 - \frac{b}{2}\right)\right)t + \sigma Z_t\right)} - \frac{a}{b}.$$

Noting that $\left(r - \frac{1}{2}\sigma^2\right)t + \sigma Z_t = \ln\frac{S_t}{S_0}$ we obtain

$$q_t = \left(q_0 + \frac{a}{b}\right)\left(\frac{S_t}{S_0}\right)^{-b} e^{b\left(r + \frac{1}{2}\sigma^2(1-b)\right)t} - \frac{a}{b}.$$

Finally

$$L_t = \left(q_0 + \frac{a}{b}\right)S_0^b S_t^{1-b} e^{b\left(r + \frac{1}{2}\sigma^2(1-b)\right)t} - \frac{a}{b}S_t.$$

When $b=0$ we have $dq = -a\left\{\left(\mu - r - \sigma^2\right)dt + \sigma dZ\right\}$ with solution

$$q_t = q_0 + e^{-a\left(\left(\mu - r - \sigma^2\right)t + \sigma Z_t\right)} = q_0 + \left(\frac{S_t}{S_0}\right)^{-a} e^{a\left(\left(r + \frac{1}{2}\sigma^2\right)t\right)},$$

and

$$L_t = L_0 + S_0^a S_t^{(1-a)} e^{a\left(\left(r + \frac{1}{2}\sigma^2\right)t\right)}.$$