LONG-TERM EXTREMES OF SOUTH AFRICAN FINANCIAL AND ECONOMIC VARIABLES

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ABSTRACT
Extreme movements in financial and economic variables are important in many actuarial applications. Despite this, there has been little research of modelling extreme outcomes over the long-term. This research investigated extremes of the core variables affecting assets and liabilities over an annual horizon. Empirical estimates of extremes and the uncertainty surrounding these were evaluated for a range of extreme quantiles and over different estimation periods. Parametric and empirical models of extreme risk measures were then used to determine the distributions that historically gave the most accurate forecasts of extremes. There was mixed success in finding models that performed well. This suggested that selecting models for predicting long-term extremes requires caution.

KEYWORDS
Financial variables; extremes; Value at Risk; backtesting

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1. INTRODUCTION

1.1 Background

Investment and economic variables affect the assets and liabilities of individuals, financial institutions, companies and governments. These variables have uncertain outcomes, including the potential for extreme outcomes which occur with low probability but can have large adverse impacts. Stochastic models forecasting the distributions of future outcomes can be used to model the long-term behaviours of these variables.

Extreme outcomes are relevant in many applications for which stochastic modelling is used. Such applications include capital modelling (Berger, Herring & Szegö, 1995), valuing options and guarantees (Wilkie, Waters & Yang, 2003), asset allocation (Lucas & Klaassen, 1998), pension funding levels (Pino & Yermo, 2011) and long-term corporate risk management (Kim & Mina, 2000). In addition, proposed insurance regulations dictate risk management based on extreme outcomes. The European Union's Solvency II,[1] the United Kingdom's Internal Capital Assessment Standards (ICAS)[2] and Solvency Assessment and Management Standards (SAM)[3] in South Africa will specify a requirement for insurers to have sufficient capital at a 99.5% confidence interval to meet all their obligations in a one-year period.

Wilkie (1986), in the description of his model, identified extreme outcomes as an aspect that required investigation and Thomson (2001) called for investigating extremes in the modelling of assets and liabilities. In practice failure to take account of extremes in investment modelling was a concern raised by the Morris review (2005), in which a criticism was that actuaries failed to give sufficient consideration to the likelihood or consequences of large adverse interest rate and equity movements. Despite this, there has been little research of extreme outcomes in the context of long-term investment modelling.

1.2 Problem Statement

This research was an investigation into the long-term modelling of extreme outcomes of South African financial and economic variables. The research questions addressed were:

— What does historical data suggest about extremes of different variables and how much uncertainty exists in this?

— What commonly-used models are suitable models to predict long-term extremes?

1.2.1 Measuring Extreme Outcomes

The approach to measuring extreme outcomes was based on measuring extreme quantiles. This is equivalent to estimating the Value at Risk (VaR) at very low probability

1 European Commission (2008). Draft Solvency II Directive
2 Financial Services Authority (2004). Interim Prudential Sourcebook
3 Financial Services Board (2011). Solvency Assessment and Management Roadmap

levels. VaR is the potential change in a variable at a given confidence level. Its use is described by Duffie and Pan (1997).

VaR can be defined as:

$$VaR = F^{-1}(p) \tag{1.1}$$

for a confidence level, p and return distribution function, F.

The measure has been shown to have deficient properties, including in its sub-additive aggregation of different risks (Artzner at al., 1999). Nevertheless, its ease of communication, prominence in regulations and proliferated use were considered to be practical advantages to using VaR in measuring extreme outcomes.

1.2.2 Definition of the time interval

In contrast to banking applications in which risks are often measured over a daily horizon, many actuarial applications require a long-term horizon. This research adopted a time horizon of one year. This may be seen as short in relation to longer time horizons that are present in life insurance and pensions. However, a yearly horizon can be useful for annual investigations or decisions, such as appraising annual changes in surplus, evaluating solvency on an annual basis and setting investment policy. Furthermore, the same methods that were used in this research can be applied to longer time horizons, given sufficient data.

A one-year horizon is consistent with the time-interval in many published stochastic models for actuarial use, such as Wilkie (1986), Thomson (1996) and Thomson and Gott (2006, 2009).

1.3 Context of research

This research investigated extremes of individual financial and economic variables. Assessing extremes in applications involving assets and liabilities may require the integration of several variables and factors. This research provided a basis for this by analysing extremes of the core variables involved.

Jointly modelling variables requires modelling the dependence of variables. Embrechts, McNeil and Straumann (1999) showed that dependence between variables can be substantially different in extreme circumstances compared to more moderate circumstances. This is not considered in this research and is the subject of further research.

The remainder of this paper is structured as follows. Section 2 describes the literature of extremes in finance and in long-term financial modelling. It also provides benchmarks for extreme outcomes in South African financial series. Section 3 outlines the data that were used in this research. Sections 4 and 5 contain the methodology and results of analysing extremes using both non-parametric and parametric methods.

Section 6 contains a summary of the research and a discussion of the results.

2. LITERATURE REVIEW

2.1 Extremes in finance

Popular financial models, such as Modern Portfolio Theory, CAPM, APT and the Black Scholes option pricing theory assume a normal distribution of returns. Mandelbrot (1963) and Fama (1965) showed that financial return distributions are leptokurtic, exhibiting high peaks and fat tails. This results in risk measures for extreme movements that are underestimated by the normal distribution, especially at very high quantiles.

There has been much subsequent research devoted to analysing and modelling extreme returns in finance. This has mainly been at high-frequency time intervals, such as daily, stemming from requirements in the banking sector (Basel Committee, 1996). The presence of extremes has been documented for various financial time series, such as equities (Danielsson & de Vries, 2000; McNeil & Frey, 2000), foreign exchange rates (Huisman et al., 2001), interest rates (Borkovec & Klüppelberg, 1998), as well as in applications such as derivatives (Cotter, 2005). Extremes of South African equities were shown by Seymour and Polakow (2003).

2.2 Long-term extremes

In contrast with short-term extremes, the literature for long-term extremes is scanty, with little explicit investigation of extremes of financial and economic distributions over the long-term.

The Benchmarking Stochastic Models working party of the Institute of Actuaries was set up with the aim of gaining a better understanding of methods that could be used for modelling extreme movements for use in capital modelling. Its work was reported in Frankland et al. (2009). The paper described the data and methods that were used to model equity returns over a one-year period. The moments of empirical data of equities were analysed, as well as how these varied between countries and different time horizons. Extreme market events were estimated based on percentiles and the estimation errors involved in deriving these were assessed using confidence intervals derived by bootstrapping. A range of parametric distributions were also fitted, with estimates of extreme percentiles calculated for each. The lack of sufficient data meant that prior distributions were assumed, without empirical justification. Suggestive benchmarks for the 99.5th percentile of returns over a year were given. Extremes of interest rates were modelled by stressing rates, based on quantiles of the normal distribution. The use of factor models to describe the yield curve was also suggested.

Frankland et al. (2009) described the practical difficulties and limitations of the work. The paucity of data on which the analysis was done was identified as a weakness. This resulted in sampling error and a wide range of possible results. This was dealt with by comparing different countries to gauge the amount of uncertainty among different datasets. Attempts were made to aggregate higher frequency data to a yearly interval, as well as scaling high frequency data to a yearly horizon, which was made possible by assuming a normal distribution. However, these did not enhance the ability to estimate

extreme percentiles of annual returns significantly. It was found that the assumption of independence of returns was problematic and it was suspected that volatility clustering was present over an annual horizon, although this was not investigated.

Embrechts, Kaufmann and Patie (2005) estimated measures of market risk over a one-year horizon using expected shortfall as a risk measure. They tested a range of models, and overcame a lack of yearly data by calibrating the models on higher frequency data and deriving annual risk measures from the monthly estimates using time-aggregation properties of the models they fitted. They compared the models' relative successes using backtesting and found that models of monthly data, scaled up to an annual horizon, were successful in their forecasts. They found that, in general, the Random Walk model performed better on average than other models for equities, exchange rates and bonds, but to varying extents.

Several published long-term stochastic investment models designed for actuarial use have attempted to model fat tails. Wilkie (1995) refined his original model for inflation by introducing heteroskedacity through the use of an ARCH model to model residuals. This resulted in unrealistically large extremes for both high and low outcomes. Hibbert, Mowbray and Turnbull (2001) formulated the equity component of their model with the objective of taking account of fat tails in equity returns. They modelled their equity model as a regime switching normal distribution. However this model did not capture the magnitude of drops implied by option prices.

2.2.1 Benchmarks for Long-term Extremes

Literature referring to extreme returns for South Africa over a one-year period could not be found. Long-term stochastic investment models built for actuarial use implicitly model extreme quantiles in their distributions and these may be the only available published figures to use in a review of possible points of reference. It must be noted that the bases of different models' estimates differ due to their estimations on different sets of data, making the estimates incomparable. The Financial Services Board (2012) provided its own benchmarks for extreme returns on equities, based on MSCI indices.

Figures 2.1 to 2.4 show estimates of a range of extreme quantiles for equities, inflation, short-term interest rates and long-term interest rates, derived from the models of Thomson (1996), Maitland (2010) and Thomson and Gott (2006), as well as the estimates of the FSB (2012) for equities.

The stochastic models were reproduced by performing 10 000 simulations of each of the models, with parameters used as published. Extreme quantiles were taken at year 10 of the simulations. It was expected that distribution would become stable through time by this point. The figures produced can thus be interpreted as extreme outcomes that could be expected in each of the variables over a one-year period without being conditional on any starting value of the simulations. Equities, short-term and long-term interest rates are shown in nominal rather than real terms and the returns on equities are the total return. The total return of equities wwa constructed from Thomson (1996) as the sum of dividend growth and the dividend yield.

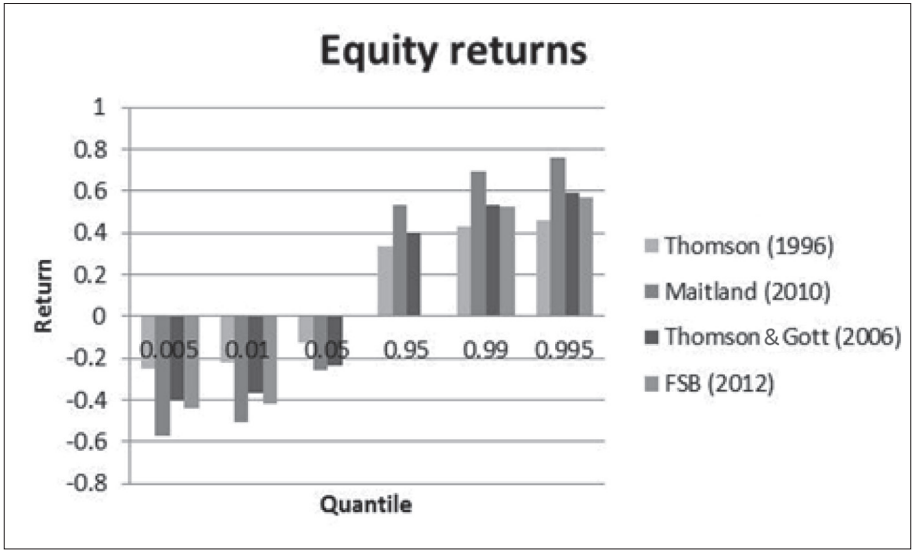

FIGURE 2.1 Extreme quantile estimates for equities

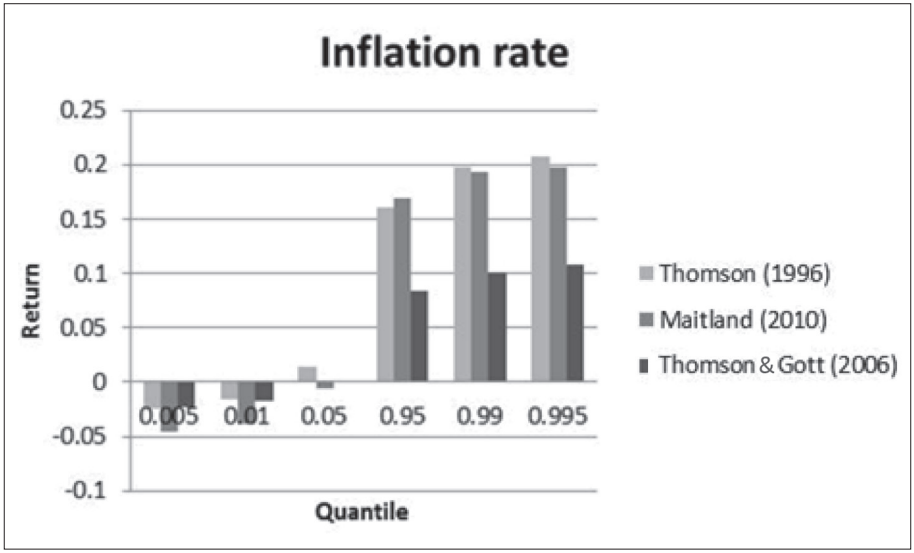

FIGURE 2.2 Extreme quantile estimates for inflation

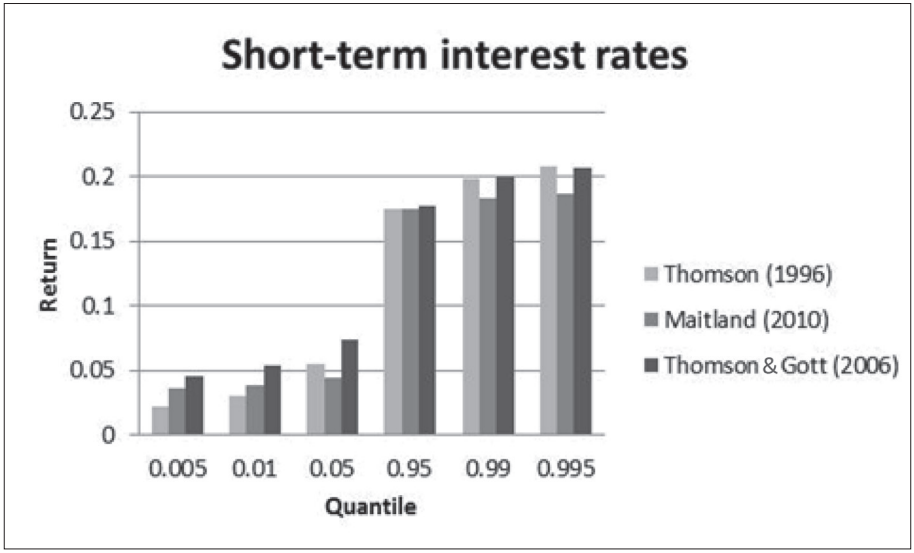

FIGURE 2.3 Extreme quantile estimates for short-term interest rates

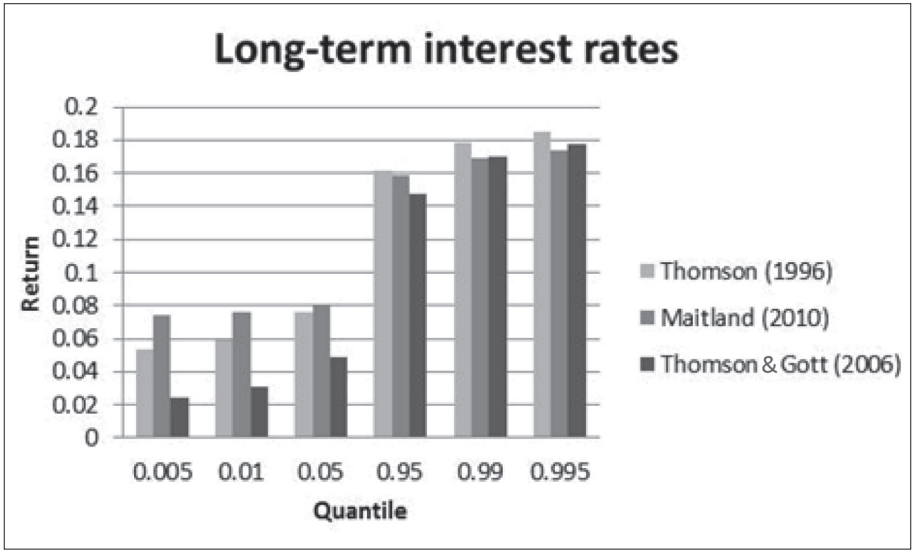

FIGURE 2.4 Extreme quantile estimates for long-term interest rates

3. DATA

3.1 Variables

The variables modelled in this research were equities, short-term interest rates, long-term interest rates and inflation. These were considered to be the core variables affecting assets and liabilities and were the variables modelled by Maitland (2010), Thomson (1996), Thomson and Gott (2006, 2009), Wilkie (1986) and other stochastic models for actuarial use. Other variables, including commodities, currencies, and alternative asset classes could potentially be modelled too. Thomson (1996) included property in his model. However, Booth (1997) cautioned the reliability of modelling the tails of property returns in actuarial models.

3.2 Data sources

A key challenge in the analysis was obtaining sufficient data to be able to estimate and validate the models that were fitted. This was found to be the main hindrance in the analysis of Frankland et al. (2009). Historical datasets going back as long as possible were thus sought.

Data series from Inet Bridge and McGregor BFA were found to have histories starting from an earliest date of 1960 for equities, inflation rates and certain short- and long-term bond yields. Firer and McLeod (1999) reported on and analysed the monthly performances of equities, bonds, cash and inflation over the period 1925 to 1998. Their dataset was compiled by blending appropriate data from a variety of sources to provide continuous series. The returns on equities were formulated as total returns. Bond data were taken from long-term bond returns up to 1979 and from the entire bond index thereafter. The return on cash was compiled by combining series of returns on three-month fixed deposits and negotiable certificates of deposit. Inflation was measured in terms of CPI, as published in official statistics. Firer and Staunton (2002) extended the series to include annual data dating back to 1900 by blending further data sources. Firer and Staunton (2002) claimed their data to be a solid historical platform on which to base estimates.

There is debate as to whether long histories of data are relevant for projections. Maitland (2010) argued that data may come from different regimes, such as the sustained period of high inflation in South Africa during the 1970s and 1980s, which may be irrelevant given South Africa's current inflation targeting policy. Conversely, Ibbotson and Sinquefield (1976) and Firer and McLeod (1999) argued that no period in history will be repeated and long histories of data contain patterns that may repeat themselves. Furthermore, extremes, as will be defined in section 3.3, may not be less prevalent under the different regimes argued by Maitland (2010). This research followed the approach of using all available historic data, provided the data were suitable for modelling. Nevertheless different periods within the datasets were tested as part of the analysis to see if they gave consistent results with the full set of data, provided that the quantity of data permitted this.

3.3 Datasets used

Data was taken from Inet Bridge for the years 1960 to 2011. The same series used by Maitland (2010) were drawn on. Data for years pre-1960 were taken from the datasets of Firer and McLeod (1999) for monthly data and Firer and Staunton (2002) for annual. These datasets were checked for consistency with the Inet data by comparing the returns of both of the data sources post-1960 and were found to be consistent.

Monthly datasets for short- and long-term interest rates were shortened because of issues with the data. Short-term interest rates remained constant for long-stretches of months in the years up to 1960. Furthermore, short-term interest rates remained at zero for extended periods during the 1930s and 1940s. According to Firer and McLeod (1999) this was due to excessive liquidity in the money market. Similarly long-term interest rates remained unchanged for long periods in the years up to the 1970s. Maitland (2010) explained that this was as a result of an illiquid bond market and the absence of an active secondary market for long-term bonds. These features would have significant effects on the analysis but are unlikely to be repeated in future. Therefore data for these periods were excluded from the monthly datasets.

Data was taken at both monthly and annual intervals, as both would be needed as part of the analysis. Tables 3.1 and 3.2 summarises the datasets that were used.

Table 3.1 Summary of monthly data sources

	1960–2011	**1925–1960**
Equities	1995–2011: JSE Total return index	Firer and McLeod (1999)
	1960–1995: JSE All share index with dividends allocated evenly over the year	
Inflation	CPI	1940–1960: Firer and McLeod (1999)
Short-term interest rates	1960–1986: 0-year bond yield	–
	1986–2011: Alexander Forbes Money-Market Index	
Long-term interest rates	1970–2011: 20-year bond yield	–

Table 3.2 Summary of annual data sources

	1960–2011	**1900–1960**
Equities	1995–2011: JSE Total return index	Firer and Staunton (2002)
	1960–1995: JSE All share index with dividends allocated evenly over the year	
Inflation	CPI	Firer and Staunton (2002)
Short-term interest rates	1960–1986: 0-year bond yield	Firer and Staunton (2002)
	1986–2011: Alexander Forbes Money-Market Index	
Long-term interest rates	20-year bond yield	1925–1960: Firer and McLeod (1999), derived from monthly yields

3.4 Differencing of data

Observations can be considered extreme either conditional on past observations or unconditionally. The approach taken was to model inflation, short-term interest rates and long-term interest rates conditional on their values at the start of a period. This amounted to modelling changes in these variables over a period rather than the levels of the variables and was achieved by taking the first differences of variables in successive periods. Equity data were formulated as annual returns.

The differencing of the interest and inflation rates should help in eliminating non-stationarity in the series. This was felt to be important since Maitland (1997) found that long-term South African financial data were non-stationary.

The differenced short- and long-term interest rates also allow for the term structure of changes in interest rates to be successfully modelled, as shown by Maitland (2002). The short-term rate can be interpreted as a factor representing changes in the level in the differenced yield curve and the long-term rate, the change in slope.

3.5 Description of data

The moments of the annual data are shown for two different estimation periods in Table 3.3. The means and standard deviations of all variables appear relatively constant over different periods. Skewness is negative for equities returns and changes for short-term interest rates and inflation, while changes in long-term interest rates have positive skewness. With the exception of equity returns, the excess kurtosis is positive for both estimation periods for all variables. The kurtosis of inflation is remarkably high for the 1900 to 2011 period but much less so for more recent estimation. This is due in large part to a single large decrease in inflation that occurred in 1920.

Table 3.3 Summary of annual data sources

	Equities		Inflation	
	1900–2011	1960–2011	1900–2011	1960–2011
Mean	9.73%	11.98%	0.01%	0.09%
Standard deviation	20.64%	23.28%	5.18%	3.51%
Skewness	–0.0876	–0.5680	–2.3891	–1.0451
Excess kurtosis	0.3318	–0.2097	15.6872	2.3015
	Short-term interest rates		**Long-term interest rates**	
	1900–2011	1960–2011	1900–2011	1960–2011
Mean	0.02%	0.06%	0.05%	0.07%
Standard deviation	2.08%	2.91%	1.08%	1.36%
Skewness	–0.6495	–0.5309	0.8317	0.6474
Excess kurtosis	6.4842	2.4924	3.5336	1.2697

The changes in skewness and kurtosis over different estimation periods indicate that different periods may have different distributions, in particular with respect to the thickness of the tails of variables. The lower kurtosis over the more recent estimation periods may result in less frequent extremes in the data. This suggests that investigations based on different estimation periods should be performed.

4. EMPIRICAL ANALYSIS

4.1 Introduction

Analysis of extremes of the four variables was first done using the empirical distributions of the annual data and not assuming any parametric distributions. An advantage of empirical analysis was that uncertainty could be estimated using a bootstrapping procedure. Empirical extremes based on a range of quantiles were analysed, as well as the effect of using different estimation periods, as suggested by section 3. The estimation periods considered were likewise 1900 to 2011 and 1960 to 2011.

4.2 The bootstrapping procedure

Bootstrapping involves generating new samples from the empirical distribution of the data by simulating observations drawn from the empirical distribution. This can enable more information to be derived from a small dataset and allows for confidence intervals to be calculated without making any assumptions about the distribution. 10 000 independent bootstrapped samples were generated of the same size as the original samples for each of the variables and for each of the estimation periods and statistics of the quantiles were then measured based on the 10 000 samples.

4.3 Quantile estimates

The empirical quantile estimates of the annual data are shown in Table 4.1.

The figures were calculated based on the median values of the bootstrapped samples, using linear interpolation to estimate percentile values when the percentiles were between data observations. The analysis was done for quantiles of both negative and positive values.

TABLE 4.1 Empirically estimated extreme quantiles

	0.005		0.01		0.05	
	1900–2011	1960–2011	1900–2011	1960–2011	1900–2011	1960–2011
Equities	–42.1%	–46.0%	–34.8%	–41.5%	–24.9%	–28.1%
Short-term interest rates	–7.4%	–9.0%	–5.0%	–7.7%	–3.2%	–3.8%
Long-term interest rates	–2.3%	2.5%	–2.0%	2.5%	–1.7%	2.5%
Inflation	–22.0%	–11.1%	–12.4%	–11.1%	–6.8%	–11.1%

	0.995		0.99		0.95	
	1900–2011	1960–2011	1900–2011	1960–2011	1900–2011	1960–2011
Equities	63.7%	53.0%	54.8%	50.2%	39.1%	40.7%
Short-term interest rates	6.7%	7.0%	6.1%	6.8%	2.9%	4.2%
Long-term interest rates	3.8%	3.9%	5.4%	3.7%	1.3%	2.5%
Inflation	13.6%	6.4%	13.0%	6.1%	5.6%	6.1%

4.4 Estimating the uncertainty

4.4.1 Extent of the Uncertainty

Confidence intervals for the quantile estimates in Table 4.1 are shown in Table A1 in Appendix A.

The confidence intervals for equities were of a range of about 20% for both the positive and negative 0.5% and 1% quantiles. The interval for the 5% quantile was narrower. Quantiles for changes in short-term interest rates had confidence intervals with ranges of about 6%. The confidence intervals of positive changes in short-term interest rates were narrower than those of the negative changes. Similarly to equities the interval for the 5% quantile was narrower than for the more extreme quantiles. Confidence intervals for extreme changes in long-term interest rates were considerably narrower than for short-term interest rates, with a range of about 1% to 2% for both negative and positive changes. Confidence intervals for negative changes in inflation rates for the 1900 to 2011 period were made wide by the single large decrease in inflation in 1920. The confidence intervals for positive extreme changes in inflation were large for the 1900 to 2011 period, but narrower the 1960 to 2011 period.

4.4.2 Differences over Estimation Periods

Intervals for extreme negative equity returns were largely similar for different estimation periods. However, for positive returns, the ranges for the 1900 to 2011 period were wider than for the 1960 to 2011 period. The pattern for long-term interest rates was the same as for equities. Changes in long-term interest rates had similar confidence intervals for both estimation periods for both positive and negative changes. Intervals for inflation were significantly wider for the 1900 to 2011 period. The change in 1920 may have been responsible for the very wide interval for negative changes in inflation. In addition to the span of the intervals, the intervals for inflation were at considerably higher levels for 1900–2011 period compared to the 1960 to 2011 period. This agrees with the observation made in section 3 that different estimation periods may produce different results.

5. PARAMETRIC ANALYSIS

5.1 Introduction

Parametric distributions have the advantage that they can extrapolate outside of what is in the dataset and can also be less prone to volatile estimates than empirical estimates when there is limited data. Furthermore, parametric distributions can exhibit properties that include fat tails, auto-regression and volatility clustering. These properties were expected to be significant for estimating extreme events.

5.2 Notation

Let:

$s_{t,E}$ denotes the equity index at time t;
$s_{t,I}$ denotes the inflation rate at time t;
$s_{t,S}$ denotes the short-term interest rate and time t;
$s_{t,L}$ denotes the long-term interest rate and time t;

In this section that follows, s_t will generically denote each of the above variables at time t as the same models will be fitted to each of the variables.

Let $r_t = s_t - s_{t-1}$ denote the changes in each of the variables from $t-1$ to t for interest rates and inflation and let $r_t = \frac{s_t}{s_{t-1}} - 1$ for equities. Models were fitted to r_t, meaning that equities were modelled as annual returns and inflation and interest rates were modelled as the absolute changes in rates over successive periods. It was assumed that r_t as strictly stationary in its distribution.

5.3 Summary of the modelling process

The modelling process consisted of fitting parametric distributions and backtesting their VaR predictions. Backtesting was recommended as a method to demonstrate the appropriateness of models for Solvency II.[4]

The backtesting procedure consisted of the following steps:

— Estimate the models based on a window of half of the points of the dataset, starting from the first observation
— Estimate the VaR over one period based on the models
— Compare the actual observations in the year following the estimation to the VaR and note whether the actual observations exceed the VaR level
— Shift the dataset forward by one period, re-estimate the models and test the VaR. Repeat this until the dataset cannot be shifted forward anymore
— Sum the number of violations for each model over all estimations and compare this to the number expected if the models were theoretically correct
— The most appropriate models for extremes can be chosen based on the proximity of the actual number of violations to the expected number.

4 CEIOPS (2008) Advice for Level 2 Implementing Measures on Solvency II: Technical provisions Article 86a. Actuarial and statistical methodologies to calculate the best estimate

5.4 Models

The models considered in this research exhibit different properties that may be relevant in predicting extremes. In addition, the empirical distribution was considered.

5.4.1 Random Walk

The Random Walk model models increments in variables over successive time periods as being independently and identically normally distributed.

The process is defined by:

$$s_t = s_{t-1} + r_t, \tag{5.1}$$

with $r_t \sim N\left(0, \sigma^2\right)$ iid for all $t \in \mathbb{N}$

The VaR estimate for the Random Walk is given by:

$$\mathrm{VaR} = \mu + \sigma x, \tag{5.2}$$

where x is the quantile of a standard normal random variable at the specified confidence level, μ is the one-period mean return up to time t, and σ the one-period standard deviation. μ and σ were estimated as effective yearly changes. Successive values of the series are assumed to be independent and so are not conditional on past observations.

5.4.2 Autoregressive Model

An autoregressive process models variables as being dependent upon previous observations of the process. The lag determines the number of previous observations on which the current observation depends. The lag was allowed to vary in the backtesting process and could be different for successive estimations.

The AR(p) process is defined by:

$$s_t = \sum_{i=1}^{p} a_i s_{t-i} + \varepsilon_t \text{ for } t \in \mathbb{N} \tag{5.3}$$

where the a_i are the autoregressive coefficients, $\varepsilon_t \sim N\left(\mu t, \sigma^2\right)$ and σ^2 is the variance of the innovation process. The VaR is given by the same formula as for the Random Walk, with μ being determined recursively and conditional on past values.

5.4.3 Stochastic Volatility Model

The Generalised Autoregressive Conditional Heteroskedastic (GARCH) model, proposed by Bollersev (1986), models the changes in variables as a function of past levels of the variable, as well as the preceding variances. This means that observations preceded by large observations and large variances are expected to have large variances themselves.

The GARCH(1,1) process is defined by:

$$s_t = s_{t-1} + r_t \text{ for } t \in \mathbb{N} \tag{5.4}$$

where

$$r_t = \sigma_t \varepsilon_t$$

$$\sigma_t^{\ 2} = \alpha_0 + \alpha_1 r_{t-1}^{\ 2} + \beta_1 \sigma_{t-1}^{\ 2}$$

ε_t iid with mean 0 and variance 1. This research assumed ε_t to follow a t-distribution to produce innovations with fat tails.

The VaR is given by the same formula as for the Random Walk, with x being a quantile of the t-distribution instead of the normal.

5.4.4 Extreme Value Theory

Extreme value theory (EVT) is a branch of probability theory which focuses on extreme outcomes. The theory derives a limiting distribution for extreme observations from the tail of a distribution that is from a strictly stationary series. Embrechts, Klüppelberg and Mikosch (1997) give the theoretical background on which the statistical methods are based. EVT was first used in finance by Longin (1996) and was motivated by Embrechts, Resnick and Samorodnitsky (1999) as a method in risk management for insurance, reinsurance and finance. The use of EVT for measuring VaR was shown in McNeil and Frey (2000). A method of EVT is to model the excesses of observations over a high threshold. Asymptotically, the excesses converge to a Generalised Pareto Distribution (GPD) with distribution function:

$$F(x) = \begin{cases} 1 - \left(1 + \xi \dfrac{x}{\beta}\right)^{\frac{1}{\xi}} & \text{if } \xi \neq 0 \\ 1 - \exp(-x/\beta) & \text{if } \xi \neq 0 \end{cases} \tag{5.5}$$

where $\beta > 0$ and $x \geq 0$ when $\xi \geq 0$ and $0 \geq x \geq -\dfrac{\beta}{\xi}$ when $\xi < 0$.

The parameter ξ determines the heaviness of the tail of the distribution and can range from a light tailed distribution when $\xi < 0$ to heavy tailed when $\xi > 0$.

The quantiles of the GPD are given by:

$$\text{VaR} = r_{(k+1)} + \frac{\beta}{\xi}\left(\left(\frac{1-q}{\frac{k}{n}}\right)^{-\xi} - 1\right) \tag{5.6}$$

where $r_{(k+1)}$ is the $(k+1)^{\text{th}}$ order statistics and n is the number of observations in the dataset.

McNeil and Frey (2000) used 10% of the observations to lie above the threshold. The same approach was taken in this research, with k set equal to $0.1n$. The approach is static in time and values are modelled unconditionally in relation to previous values.

5.5 Data frequency for the estimation

Testing extremes over an annual horizon directly was not plausible because the number of observations was too small to perform meaningful analysis. For example, backtesting VaR using the 0.5% quartile and 110 available data points would be expected to produce 0.225 violations. This could lead to spurious analysis, with many models giving zero violations. This problem was recognised by Dowd and Blake (2006), who identified that the estimation of risk measures over long-term horizons, such as one year, involves uncertainty in estimation because of the relatively small number of annual data points that would be available. They also stated that validation is more difficult because of the lack of data.

Scaling of higher frequency data to an annual horizon, using appropriate scaling rules, can lead to more plausible models of extreme outcomes because of their more efficient use of data. This comes at the expense of bias created by the use of scaling procedures.

Frankland et al. (2009) scaled monthly data to annual assuming data followed a Random Walk. They found this to produce poor results and concluded that the independence assumption of the Random Walk was not valid. Embrechts, Kaufmann and Patie (2005) presented scaling rules for the Random Walk, AR(p), GARCH(1,1) and EVT models. This provided scaling rules for a more appropriate range of distributions compared to Frankland et al. (2009). Embrechts, Kaufmann and Patie (2005) used these methods to scale higher frequency distributions to yearly distributions. Their study showed that scaling from 1-month distribution to annual can be done successfully and produced superior forecasts to annually estimated estimates of risk measures in their study. The scaling laws for the distributions that were used are set out in Appendix B.

5.6 Results

5.6.1 Introduction

As part of the backtesting procedure, the models were estimated using monthly data and the risk measures were scaled to annual horizons using the applicable scaling laws. Models were estimated on data from the periods 1900 to 2011 and 1960 to 2011, as suggested by the data description and the empirical analysis.

5.6.2 Issues in Modelling Stochastic Volatility

Within the backtesting process the GARCH model produced parameters that were found to violate the stationarity condition, $\alpha_1 + \beta_1 < 1$. This was similar to what was found in the analysis of Embrechts, Kaufman and Patie (2005). The GARCH model was thus excluded from the analysis and stochastic volatility was therefore not tested for in the data.

It should be noted that Christofferson, Diebold and Schuermann (1998) found that over long-term horizons the effect of stochastic volatility is short-lived. They suggested that after a horizon of 10 days stochastic volatility wears off and that forecasting stochastic volatility is not plausible.

5.6.3 Backtesting Results

The results of the backtesting are shown in Table A2 in Appendix A. The results are presented as the percentage of backtesting runs for which violations occurred. Equities returns and changes in inflation were estimated over two different estimation periods as the amount of data allowed this.

The results show the Random Walk and empirical estimation to be the most successful models for both negative and positive quantiles of equities. The success of the autoregressive model was inconsistent among different estimation periods for equities. Extremes of changes in short-term interest rates were predicted most successfully by the Random Walk and autoregressive models, with the Random Walk the best for negative changes in rates and the autoregressive model for positive changes. The best models for changes in long-term interest rates were identical to those for short-term interest rates. The Random Walk was the best performing model for both positive and negative extreme changes in inflation.

These results were tested for significance. The number of violations should follow a binomial distribution, with parameters q and n, representing the theoretical probability of a violation and the number of trials, respectively. To test the null hypothesis that the respective models correctly estimate a quantile, 95% confidence intervals were calculated. If the number of violations was inside the interval, the hypothesis was not rejected and this is indicated in the table by an asterisk.

Few of the backtesting results for equities were found to be within the confidence intervals of the theoretical number of violations. Similarly negative quantiles for both short-term interest rates and inflation were not within confidence bounds. This indicates that the models were not particularly successful in forecasting these extremes.

5.6.4 Results over Different Periods and Quantiles

Equities returns and changes in inflation were estimated over two different estimation periods as the quantity of data allowed this. The best fitting models were consistent across both of the periods for both variables. However, the results for the 1960 to 2011 period were superior for equities and conversely were inferior for changes in inflation. The models that performed the best for each of the variables were consistent across the different quantiles.

6. SUMMARY AND DISCUSSION

6.1 Summary of results

Investigations were performed using empirical, as well as parametric analyses. The empirical investigation evaluated confidence intervals that indicated the level of uncertainty associated with the estimation. Different estimation periods and quantiles were evaluated. Parametric distributions with a range of properties were fitted using monthly data, and resulting risk measures were scaled to an annual time horizon. This enabled validation of the models through backtesting of forecasts. It should

be noted that the GARCH model was excluded from the analysis due to problems with model estimation. This was felt to be acceptable since Christofferson, Diebold and Schuermann (1998) found that over long-term horizons the effect of stochastic volatility is short-lived.

There was a large degree of uncertainty in extreme equity returns, indicated by the wide confidence intervals of the empirical quantiles for equities. Positive changes in short-term interest rates showed less uncertainty than negative changes, indicated by the narrower empirical confidence intervals, and long-term interest rates exhibited much less uncertainty than short-term. Uncertainty in changes of inflation appeared large. The uncertainty in estimates for the 1900 to 2011 period was often greater than for the 1960 to 2011 period.

The Random Walk and empirical estimation appeared to be the most successful models for both negative and positive extreme quantiles. However, none of the models were particularly successful in forecasting annual extremes of equity returns, as shown by the significance tests for the backtesting results. Extremes changes in short-term interest rates were predicted most successfully by the Random Walk and autoregressive models, with the Random Walk the best model for negative changes in short-term interest rates and the autoregressive model for positive changes. However, like equities, none of the models produced significantly good predictions for negative extremes. The best models for changes in long-term interest rates were found to be the Random Walk and autoregressive models, similarly to short-term interest rates. The best model for changes in inflation was the Random Walk.

The choice of best-performing model was generally robust to the estimation period. This suggests that the results of the backtesting appeared to be a function of the models tested rather than anomalous behaviour in the data. The more recent 1960 to 2011 period resulted in better predictions for equities but worse for inflation. The performances of models were also generally robust to the extreme quantiles considered. This suggests that provided that the correct modelling approach was taken, it should be sufficient to model extremes of different quantiles.

6.2 Limitations and further work

This research, like Frankland et al. (2009), assumed that the data were stationary. This allowed the use of a range of commonly-used and parsimonious statistical methods for which scaling properties were available. However, the stationarity assumption was not tested. A wider selection of models, including non-stationary models such as the Regime Switching model can be investigated. In particular, further investigation into equities is needed given the poor results for equities. However, validation of models may be difficult given the lack of historical data.

Other topics for further research include:

— Alternative measures of extremes could be used, including the use of Expected Shortfall

— Further work can be done on estimating statistical uncertainty of the parametric models used
— Dependence between extremes of different variables can be investigated

6.3 Conclusion

The empirical analysis indicated the levels of extremes and variations that have been experienced historically. The results can be used as ex-post estimates of extremes, as well as of the uncertainties in extremes. Uncertainty existed to different extents in different variables and taking this into account can be important in applications for which extremes of these variables are material.

Success in modelling annual extremes using parametric models was was mixed, as opposed to more successful modelling using higher frequency data that has been seen in literature. In the cases of equities, negative short-term interest rates and inflation rates, this research was not able to infer any particularly successful statistical models based on a range of potential models. This suggests that selecting models for predicting extremes of these returns is difficult and requires further investigation.

Given the materiality of extremes for the requirements of Solvency II and SAM and other actuarial applications, this research suggests caution in using stochastic models for theses puposes. Other methods of evaluating extremes, such as expert opinion, and deterministic approaches, such as stress testing, may also be useful in complementing stochastic modelling.

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APPENDIX A

Table A1 Bootstrapped confidence intervals

	Equities					
	0.005		0.01		0.05	
	1900	1960	1900	1960	1900	1960
0.05	–50.5%	–50.5%	–50.5%	–50.5%	–31.2%	–42.2%
0.95	–30.1%	–28.6%	–28.6%	–26.2%	–15.9%	–16.1%
	0.995		0.99		0.95	
	1900	1960	1900	1960	1900	1960
0.05	42.3%	40.7%	40.6%	40.7%	34.8%	36.9%
0.95	73.1%	55.9%	73.1%	55.9%	42.7%	50.6%
	Short-term interest rates					
	0.005		0.01		0.05	
	1900	1960	1900	1960	1900	1960
0.05	–10.2%	–10.2%	–10.2%	–10.2%	–3.8%	–7.9%
0.95	–3.8%	–3.9%	–3.5%	–3.8%	–2.5%	–2.8%
	0.995		0.99		0.95	
	1900	1960	1900	1960	1900	1960
0.05	3.9%	4.2%	3.5%	3.5%	2.2%	2.8%
0.95	7.3%	7.3%	7.3%	7.3%	4.2%	6.8%
	Long-term interest rates					
	0.005		0.01		0.05	
	1900	1960	1900	1960	1900	1960
0.05	–2.6%	–2.6%	–2.6%	–2.6%	–2.0%	–2.3%
0.95	–1.8%	–1.9%	–1.8%	–1.8%	–1.5%	–1.7%
	0.995		0.99		0.95	
	1900	1960	1900	1960	1900	1960
0.05	2.5%	2.9%	1.6%	2.4%	1.0%	1.2%
0.95	3.9%	3.9%	3.9%	3.9%	3.5%	3.8%

	Inflation					
	0.005		0.01		0.05	
	1900	1960	1900	1960	1900	1960
0.05	–33.3%	–12.5%	–33.3%	–12.5%	–11.1%	–12.5%
0.95	–9.4%	–6.0%	–7.1%	–6.0%	–5.0%	–6.0%
	0.995		0.99		0.95	
	1900	1960	1900	1960	1900	1960
0.05	7.8%	–12.5%	6.4%	–12.5%	4.3%	3.7%
0.95	13.6%	–6.0%	13.6%	–6.0%	8.8%	6.1%

Table A2 Backtesting results

Equities						
	0.005		0.01		0.05	
	1925	1960	1925	1960	1925	1960
Random Walk	4.1%	1.4%	6.1%	2.7%	12.4%	8.8%
Autoregressive	9.2%	9.9%	10.6%	12.2%	14.3%	18.4%
EVT	15.1%	5.1%	16.9%	6.8%	22.2%	14.3%
Empirical	2.9%	0.0% *	4.7%	0.7% *	12.2%	8.2%
Expected	0.5%	0.5%	1.0%	1.0%	5.0%	5.0%
	0.995		0.99		0.95	
	1925	1960	1925	1960	1925	1960
Random Walk	5.5%	0.0% *	6.5%	0.0% *	14.3%	1.7%
Autoregressive	7.3%	1.7%	10.4%	3.1%	23.9%	11.9%
EVT	54.5%	63.6%	57.6%	64.6%	62.7%	66.3%
Empirical	5.5%	0.0% *	7.6%	0.0% *	15.3%	2.7% *
Expected	0.5%	0.5%	1.0%	1.0%	5.0%	5.0%
Short-term interest						
	0.005		0.01		0.05	
		1960		1960		1960
Random Walk		3.7%		4.8%		11.2%
Autoregressive		4.1%		5.4%		14.6%
EVT		12.2%		17.0%		31.6%
Empirical		1.7%		2.7%		11.9%
Expected		0.5%		1.0%		5.0%
	0.995		0.99		0.95	
		1960		1960		1960
Random Walk		0.7% *		2.0% *		5.1% *
Autoregressive		0.3% *		0.3% *		2.4%
EVT		4.8%		7.5%		18.7%
Empirical		0.7% *		0.3% *		5.1% *
Expected		0.5%		1.0%		5.0%

Long-term interest						
	0.005		0.01		0.05	
		1970		1970		1970
Random Walk		0.9% *		1.7% *		5.1% *
Autoregressive		36.3%		41.5%		51.3%
EVT		23.1%		26.1%		39.3%
Empirical		0.0% *		1.3% *		5.1% *
Expected		0.5%		1.0%		5.0%
	0.995		0.99		0.95	
		1970		1970		1970
Random Walk		2.6%		3.4%		5.1% *
Autoregressive		0.0% *		0.0% *		2.1%
EVT		10.7%		11.5%		14.5%
Empirical		3.4%		3.0%		4.7% *
Expected		0.5%		1.0%		5.0%
Inflation						
	0.005		0.01		0.05	
	1925	1960	1925	1960	1925	1960
Random Walk	2.1%	3.4%	2.6%	3.4%	7.1%	9.5%
Autoregressive	20.0%	37.1%	21.7%	38.8%	30.7%	48.0%
EVT	36.7%	41.2%	37.6%	42.5%	43.3%	43.9%
Empirical	1.9%	3.1%	2.4%	3.4%	7.1%	8.5%
Expected	0.5%	0.5%	1.0%	1.0%	5.0%	5.0%
	0.995		0.99		0.95	
	1925	1960	1925	1960	1925	1960
Random Walk	0.7% *	1.4%	0.7% *	1.7% *	2.1%	6.1% *
Autoregressive	20.7%	3.1%	21.7%	3.7%	27.9%	8.5%
EVT	43.8%	30.6%	44.0%	33.3%	46.0%	40.1%
Empirical	1.2%	1.7%	1.4% *	1.7% *	2.6%	7.8%
Expected	0.5%	0.5%	1.0%	1.0%	5.0%	5.0%

APPENDIX B
Scaling Rules

Let r_t^k denote the log return of equity or differenced inflation rates, short term interest rate or long-term interest rate over k periods. The scaling rules that follow enable the calculation of k-period VaR estimates. In this research monthly data were used and thus k was set to 12.

B1. Random Walk

If r_t are independently and identically normally distributed with mean 0 and variance, σ_2, then $r_t^k = \sum_{i=1}^{k-1} r_{t-i}$ has mean 0 and stand deviation $\sqrt{k}\sigma$.

Thus, for independently and identically normally distributed one-period returns with mean μ and variance, σ^2, $r_t^k \sim k\mu + \sqrt{k}(r_t - \mu)$.

The one-period parameters μ and σ can be estimated by $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} r_i$ and

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \mu)^2}.$$

The k-period parameters can be used as parameters in the VaR calculation (equation 5.2).

B2. Autoregressive model

The AR(p) process can be detrended by setting $\overline{s_t} = s_t - \mu t$. The detrended AR($p$) model is $\overline{s_t} = \sum_{i=1}^{p} a_i\, s_{t-i} + \overline{\epsilon_t}$, where $\mu = \frac{\mu_1}{1-\sum_{i=1}^{p} a_i}$, $\overline{\epsilon_t} \sim N(0, \sigma^2)$ iid and μ_1 is the drift of the original process.

To calculate the k-period parameters from $\overline{s_t}$:

— Calculate $\overline{s_t}$ using $\hat{\mu} = \frac{s_n - s_0}{n}$

— Fit the AR(p) process to the $\overline{s_t}$. Maximum Likelihood estimation gives the parameters $\hat{p}, \hat{a}_i \left(i = 1, 2, \ldots, \hat{p} \right)$ and σ

— Forecast the k-period return using the relation $\hat{\mu}_k = kh\,\hat{\mu} + \hat{m}$ where $\hat{m} = \tilde{s}_{t+k} - \tilde{s}_t$ and $\tilde{s}_{t+j}$ is defined recursively by $\tilde{s}_{t+j} = \sum_{i=1}^{\hat{p}} \hat{a}_i\, \tilde{s}_{t+(j-i)}$ $(j = 1, \ldots, k)$ and $\tilde{s}_u = \overline{s_u}$ for $u < t$

— Forecast the k-period volatility using $\hat{\sigma}^k = \hat{\sigma}\sqrt{\sum_{j=0}^{k-1}\delta_j^{\,2}}$, with $\delta_0 = 1$ and $\delta_j = \sum_{i=1}^{j}\hat{a}_i\,\delta_{j-i}$,$\hat{a}_i = 0$ for all $i > \hat{p}$

Conditioned on s_t, the one year forecast s_{t+k} is distributed $N(s_t + \mu_k, \sigma^{k2})$. The k-period parameters can be used as parameters in the VaR calculation (equation 5.2)

B3. GARCH(1,1) model

Drost and Nijman (1993) showed that for a GARCH(1,1) process, under regularity conditions, the corresponding k-period process is a weak GARCH(1,1) model with

$$(\sigma_t^{\,k})^2 = \alpha_{k,0} + \alpha_{k,1}(r^k{}_{t-kh})^2 + \beta_{1,k}(\sigma^k{}_{t-kh})^2,$$

where

$$\alpha_{k,0} = k\alpha_0\frac{1-(\alpha_1+\beta_1)^k}{1-(\alpha_1+\beta_1)},\ \alpha_{k,1} = (\alpha_1+\beta_1)^k - \beta_{k,1},$$

$\left|\beta_{k,1}\right| < 1$ is the solution to the equation $\dfrac{\beta_{k,1}}{1+\beta_{k,1}^{\ 2}} = \dfrac{a(\alpha_1+\beta_1)^k - b}{a\left(1+\left(\alpha_1+\beta_1\right)^{2k}\right)-2b}$

$$a = k(1-\beta_1)^2 + 2k(k-1)\frac{\left(1-\beta_1-\alpha_1\right)^2\left(1-\beta_1^{\,2}-2\alpha_1\beta_1\right)}{(\kappa-1)\left(1-\left(\alpha_1+\beta_1\right)^2\right)} + 4\frac{k-1-k\left(\alpha_1+\beta_1\right)+\left(\alpha_1+\beta_1\right)^k)(\alpha_1-\alpha_1\beta_1(\alpha_1+\beta_1))}{1-(\alpha_1+\beta_1)^2}$$

$$b = (\alpha_1 - \alpha_1\beta_1(\alpha_1+\beta_1))\,\frac{1-(\alpha_1+\beta_1)^{2k}}{1-(\alpha_1+\beta_1)^2}$$

and κ denotes the kurtosis of the k-period return and can be worked out according to the steps given in Embrechts, Kaufman and Patie (2005).

The parameters of the one-period GARCH process can be estimated using Quasi-Maximum Likelihood based on normally distributed innovations.

At time t, the conditional volatility $\hat{\sigma}^k = \sigma\left(t,t\right)$ can be forecast using a recursive relation:

$$\hat{\sigma}^2\left(t^*,t\right) = \hat{\alpha}_0 + \alpha_1(\hat{r}_{t^*} - \mu^k)^2 + \beta_1\hat{\sigma}^2(t^*-k,t) \text{ for } t^* = t-\left(n-1\right)k\ldots t-k,t$$

and $\hat{\sigma}^2\left(t-nk,t\right) = \dfrac{k}{nk-1}\sum_{i=0}^{nk-1}(r_{t-i}-\hat{\mu})^2$. To estimate the k-period mean, $\mu_k = k\hat{\mu}$ and μ is the one-period mean return up to time t.

B4. EVT model

Dacorogna et al. (1999) showed that for a quantile, x, from the EVT distribution, $x^k \sim k^{1/\alpha} x$ as the quantile tends towards 0. This mean that increasing a one-period time horizon to k-period increases the VaR for the EVT model by a factor $k^{1/\alpha}$ where $\alpha = \frac{1}{\xi}$ and ξ is the shape parameter as in equation 5.5.

Note that for $\alpha > 2$, as was the case in many of the models fitted in the backtesting procedure in this research, the scaling factor is smaller than the normal model, where the VaR is increased by a factor $k^{1/2}$. Thus, in comparison with the normal model, the scaling factor may mean that the k-period quantile may be larger for the normal model than for the EVT model.