HOW A SINGLE-FACTOR CAPM WORKS IN A MULTI-CURRENCY WORLD

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ABSTRACT

As shown in AD Wilkie's paper entitled "Why the capital-asset pricing model fails in a multi-currency world" there is no unique single-factor capital-asset pricing model (CAPM) in a multi-currency world. As he points out, the standard CAPM assumes that all investors measure risk and return in the same currency. He shows that, if two investors measure risk and return in different currencies, the standard CAPM cannot describe the pricing of capital assets for both investors. The aim of this paper is to give actuaries a way ahead in the use of the single-factor CAPM in a multi-currency world. It assumes that, for every currency in which investors measure risk, there is a unique CAPM across all the markets in which they invest. It develops a theory for multi-currency CAPMs by developing a CAPM for each set of investors that measures their returns in a particular currency. In the development of this theory the meanings of homogeneous expectations and of equilibrium are reconsidered in the context of a multi-currency world.

In this paper a single-factor multi-currency CAPM (SFM-CAPM) is developed. It is shown that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the *ex-ante* variances and covariances of returns. The estimation of the variance–covariance matrix of returns by constrained maximum-likelihood estimation is discussed. Some difficulties with that approach are explained and an alternative approach, using ordinary least squares, is developed. The theory is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira. The application is designed for use by actuaries in the modelling of the assets and liabilities of long-term financial institutions. To that end the longest

possible range of time periods is used and quarterly intervals are used rather than the relatively short time intervals typically used in the literature. Indications are given of the way in which the findings of this paper will lead to further research for the purposes of such modelling.

KEYWORDS
International CAPM, single-factor multi-currency CAPM (SFM-CAPM)

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1. INTRODUCTION

1.1 As shown by Wilkie (unpublished), there is no unique single-factor capital-asset pricing model (CAPM) in a multi-currency world. As he points out, the standard CAPM assumes that all investors measure risk and return in the same currency. He shows that, if two investors measure risk and return in different currencies, the standard CAPM cannot describe the pricing of capital assets for both investors.

1.2 However, long-term financial institutions do invest abroad and, for the purposes of modelling liabilities in an incomplete market, it is necessary to find an equilibrium model that satisfies reasonable assumptions about homogeneous expectations and how investors measure risk and reward in a multi-currency world. The aim of this paper is to give actuaries a way ahead in the use of the single-factor CAPM in a multi-currency world for the purposes of the stochastic modelling of the assets and liabilities of long-term financial institutions. It assumes that, for every currency in which investors measure risk, there is a unique CAPM across all the markets in which they invest. It develops a theory for multi-currency CAPMs by developing a CAPM for each set of investors that measures their returns in a particular currency. In the development of this theory the meanings of homogeneous expectations and of equilibrium are reconsidered in the context of a multi-currency world.

1.3 In section 2 the literature on international versions of the CAPM is reviewed. In section 3 a single-factor multi-currency CAPM (SFM-CAPM) is developed. It is shown that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the *ex-ante* variances and covariances of returns. The estimation of the variance–covariance matrix of returns by constrained maximum-likelihood estimation is discussed. Difficulties were experienced with the application of this theory. These difficulties are explained

and an alternative approach, using ordinary least squares, is developed. The theory is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira. The data obtained for this purpose and the results of the application are described in section 4. The application is designed for use by actuaries in the modelling of the assets and liabilities of long-term financial institutions. To that end, the longest possible range of time periods is used and quarterly intervals are used rather than the relatively short time intervals typically used in the literature. Section 5 concludes the paper with a summary of the findings and some indications of the way in which those findings will lead to further research for the purposes of such modelling.

2. LITERATURE REVIEW

2.1 The Domestic Capital-Asset Pricing Model and Market Segregation

2.1.1 In order to distinguish between the CAPM used in a single domestic (segregated) market and the CAPM used in an international (integrated) market, the terms 'domestic CAPM' and 'international CAPM' are used respectively.

2.1.2 A market in which the price of an asset depends on where it is traded may be referred to as a 'segregated market' (Karolyi & Stulz, 2003). If markets are segregated then different domestic capital markets can be considered as independent entities and the international market consists of individual segregated markets (Solnik, 1974b). Jorion & Schwartz (1986) pointed out that complete segregation implies that only domestic factors such as domestic systemic risk should enter the pricing of assets. Stulz (1981) stated that the widespread use, in all countries that have an equity market, of some proxy of the domestic market portfolio to determine how domestic assets are priced can be justified only by an assumption that markets are internationally segregated.

2.1.3 In terms of the traditional CAPM that uses the domestic market portfolio, which we call the 'domestic CAPM', the equilibrium expected return of an asset may be expressed as below:

$$E\{R_i\} = R_F + \beta_i\left[E\{R_M\} - R_F\right]; \tag{1}$$

where R_i, R_F and R_M are the returns on security i, on the risk-free asset F and on the domestic market portfolio M respectively. β_i represents the sensitivity of the asset return to market movements; i.e.:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_{MM}};$$

$$\sigma_{iM} = cov\left\{R_i, R_M\right\}; \text{ and}$$

$$\sigma_{MM} = var\left\{R_M\right\}.$$

2.2 The Domestic Capital-Asset Pricing Model and Market Integration

2.2.1 Karolyi & Stulz (op. cit.) defined an 'integrated market' as a market in which assets have the same price regardless of where they are traded. Therefore investors should earn the same risk-adjusted expected return on similar assets in different domestic markets, which is consistent with the definition of integration by Jorion & Schwartz (op. cit.). Jorion & Schwartz (op. cit.) argued that, with integration, the world market index should be mean–variance efficient, and as a result, the only priced risk should be systematic risk relative to the world market.

2.2.2 Because the domestic CAPM considers only domestic investment, it has important limitations (Solnik, 1974a). As Solnik (1974a) pointed out, because there is no universal risk-free asset, and because of exchange-rate risk, there is little intuitive reason to expect that the simple risk-pricing relation in the CAPM could be applied at the international level . Since then, certain authors (e.g. Stehle, 1977; Stulz, 1995a) have argued that a domestic CAPM is appropriate only for an asset that is traded in a closed, domestic financial market. Wilkie (op. cit.) showed that there is no unique CAPM in a multi-currency world. He showed that, if two investors measured risk and return in different currencies, the standard CAPM could not describe the pricing of capital assets for both investors. Karolyi & Stulz (op. cit.) argued that there are systematic patterns in ownership of foreign equities that are hard to reconcile with models assuming perfect financial markets (such as the traditional CAPM) and therefore the only way to rationalise these patterns would be to argue that the gains from international diversification are too small to make it worthwhile to hold foreign assets. The inadequacies of the traditional CAPM in an international setting have therefore led to extensive debate and the development of equilibrium models (most of which are variations of the CAPM) to incorporate exchange-rate risk and global market portfolios.

2.3 The International Capital-Asset Pricing Model

2.3.1 As pointed out by Ng (2004), the starting point of the international CAPM literature is the observation that purchasing power parity does not hold. This means that, in real terms, investors who measure their returns in different currencies earn different returns. This contravenes the standard CAPM assumption that investors have homogeneous expectations of returns, and it presents challenges for the aggregation of individual portfolios into a general asset pricing equation. Wilkie (op. cit.) also concluded that when different currencies exist the traditional CAPM's assumption that all investors measure risk in the same currency breaks down. Stulz (1981) argued that, without a model showing how assets are priced in a world in which asset markets are fully integrated, it is impossible to determine whether asset markets are segregated internationally or not. In the international CAPM (ICAPM) of Solnik (1974a), Sercu (1980) and Stulz (1981), exchange-rate risk is priced by modifying the CAPM. The ICAPM contains risk premia that are based on the covariances of assets with exchange

rates. There are different versions of the ICAPM. For the purpose of this paper, to illustrate the application of the single factor CAPM in a multi-currency world, only the single-factor and multi-factor ICAPMs are considered.

2.3.2 The single-factor ICAPM, also referred to as the 'global CAPM' (GCAPM), as developed by Solnik (1974a), Grauer, Litzenberger & Stehle (1976), Sercu (op. cit.), Adler & Dumas (1983), Stulz (1981), and others, is expressed as follows:

$$E\{R_i\} = R_F + \beta_i^w \left[E\{R_W\} - R_F \right]; \tag{2}$$

where R_i, R_F and R_W are the nominal returns in domestic currency on security i, on the risk-free asset F and on the global market portfolio W respectively. β_i^w represents the sensitivity of the asset return to global market movements; i.e.:

$$\beta_i^w = \frac{\sigma_{iW}}{\sigma_{WW}}$$

$$\sigma_{iW} = cov\left\{R_i, R_W\right\}; \text{ and}$$

$$\sigma_{WW} = var\left\{R_W\right\}.$$

Thus the GCAPM looks at the world from the perspective of one currency only—the currency in which the investor measures returns on the risk-free asset and on the global market portfolio. If strict purchasing-power parity applies, and if returns are measured in real terms (or there is no inflation), then equation (2) applies regardless of the currency in which the investor measures returns. Under such circumstances the risk-free rate in each currency is equal to the risk-free rate in every other currency, the mean–variance optimal portfolio for each investor is equal to that of every other investor, regardless of the currency in which they measure returns, and the variances and covariances of returns—and therefore the beta of every asset—is similarly the same for every investor. (cf. e.g. Karolyi & Stulz, op. cit.)

2.3.3 In an attempt to determine the factors that affect share-price movements across the world, Solnik (1974b) determined the international market structure of asset prices. The resulting model is referred to as the 'multi-factor ICAPM'. The risk-pricing relation for the multi-factor ICAPM for asset i may be expressed as follows:

$$E\left\{R_i\right\} = R_F + \beta_i^w \left[E\left\{R_w\right\} - R_F \right] + \gamma_i^1 \left[E\left\{R_M^1\right\} - R_F^1 \right] + \gamma_i^2 \left[E\left\{R_M^2\right\} - R_F^2 \right] + \\ \ldots + \gamma_i^C \left[E\{R_M^C\} - R_F^C \right]; \tag{3}$$

where R_i, R_F, R_W R_M^c and R_F^c are the returns on security i (in domestic currency), on the risk-free asset F (in domestic currency), on the global market portfolio W, on the

market portfolio in currency *c* and on the risk-free asset in currency *c* respectively. β_i^w represents the sensitivity of the asset return to global market movements. γ_i^1 to γ_i^C are the sensitivities of asset *i* to the currencies 1 to *C* (the number of exchange rate factors can be as many as the number of currencies other than the numeraire currency); i.e.:

$$\beta_i^w = \frac{\sigma_{iW}}{\sigma_{WW}};$$

$$\sigma_{iW} = cov\{R_i, R_W\}; \text{ and}$$

$$\sigma_{WW} = var\{R_W\}.$$

Under the assumptions of this model, a risk-free domestic asset is not risky for a domestic investor, but, because of currency risks, this same risk-free domestic asset is a risky asset for a foreign investor. The sensitivity parameters included in the ICAPM model capture these risks. Once again, that model looks at the world from the perspective of the currency in which the investor measures returns. Furthermore, there is no reason why, from the point of view of an investor who measures returns in that currency, investments in other currencies should not be diversified across currencies, nor why, for such an investor, the risks of investment in another currency should not be priced consistently with the risks of investment in the domestic currency.

2.3.4 Karolyi & Stulz (op. cit.) demonstrated that systemic mistakes are possible when one uses the domestic CAPM and when domestic investors have access to international markets. Other authors have demonstrated pricing errors from using the domestic CAPM in an integrated market such as Dolde et. al. (2011), Stulz (1995b), Stulz (1995c) and Koedijk et. al. (2002).

2.3.5 It is standard practice in the USA to use a long-term (say 30-year) yield on treasury bills as a proxy for the risk-free rate (Stulz, 1995c). However, Stulz (1995c) uses monthly returns and computes arithmetic means of returns per period over a long sample period during which markets were fairly integrated. Jorion & Schwartz (op. cit.) derive the risk-free rate from the yield on three-month treasury bills and use monthly returns. In principle one should measure the risk-free rate over a time interval up to the assumed time horizon of market participants, or the interval after which portfolio selection will be reconsidered. This time interval should be equal to the unit time interval adopted. It is inevitable that risk-free rates will have varied over the sample period. Unless one applies the CAPM using time series, the choice of a risk-free rate is problematic, even in a domestic CAPM.

2.3.6 As observed above, Wilkie (op. cit.) showed that there is no unique CAPM in a multi-currency world. He showed that, if two investors measured risk and return in different currencies, the standard CAPM could not describe the pricing of capital assets for both investors. It is accepted in this paper that the assumption of strict

purchasing-power parity is untenable, so that the GCAPM is untenable. Furthermore, it is assumed in this paper that exposure to variances of returns arising from currency exposure is no different from exposure to variances of returns arising from domestic sources, so that a multi-factor model is unnecessary.

2.3.7 In this paper, in the light of Wilkie's (op. cit.) conclusion and the literature reviewed, the authors develop a theory for a single-factor CAPM in a multi-currency world. They do so by specifying a CAPM for each set of investors that measures their returns in a particular currency. They assume that, for every currency in which investors measure risk, there is a unique CAPM across all the markets in which they invest.

3. THE NECESSARY AND SUFFICIENT CONDITION FOR THE SINGLE-FACTOR MULTI-CURRENCY CAPM

In this section a single-factor multi-currency CAPM (SFM-CAPM) is developed. Section 3.1 is a preliminary discussion largely devoted to the definitions required for the following sections. In section 3.2 it is shown that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the *ex-ante* variances and covariances of returns. The SFM-CAPM is formulated in section 3.3. Difficulties were experienced with the application of constrained maximum-likelihood. In section 3.4 these difficulties are explained and alternative approaches are considered. One of these approaches, which uses ordinary least squares, is developed in section 3.5.

3.1 Preliminary Discussion

3.1.1 Suppose there are C currencies and that, in currency c, there is one risk-free asset and n_c risky capital assets have been issued. It is assumed that every investor measures investment returns in one of these currencies. Regardless of the currency in which an investor measures investment returns, the investor may invest in any currency. An 'asset issued in currency c' is a risky asset issued in that currency or the risk-free asset denominated in that currency. (For an investor who measures returns in another currency, the risk-free asset denominated in currency c is not risk-free; this matter is dealt with in greater detail below.) The 'return in currency c' on an asset issued in currency d is the force of return earned on that asset, over a unit interval, measured in currency c. Whilst it is customary to measure returns as rates, there are substantial advantages to the use of forces. The implicit assumption of this approach is that assets are continuously rebalanced during the unit interval, so that the weightings of the respective forces remain constant. Exchange rates are measured per unit of currency 1. The increase in the exchange rate of currency c per unit of currency 1 is measured as a force of increase over the unit interval. Returns and increases in exchange rates may be measured in real terms (relative to a price index) or in nominal terms.

3.1.2 We assume that the CAPM applies for investors in each currency. More specifi-

cally, we assume that:

1. investors who measure their investment returns in currency *c* (i.e. 'currency-*c* investors') have indifference curves in mean–variance space, the means and variances being those measured in that currency; and
2. investors have homogeneous expectations of the means, variances and covariances of returns in each currency on assets issued in that currency.

3.1.3 First we consider returns in currency *c* on assets issued in that currency. For this purpose we define the following random variables, where, for $c = 1, \ldots, C$, $i = 1, \ldots, n_c$ denotes the risky assets issued in that currency:

— X_{ci} is the return in currency *c* on risky asset *i* issued in that currency for $c = 1, \ldots, C;\ i = 1, \ldots, n_c$; and
— X_c is the increase in the exchange rate of currency *c* per unit of currency 1 for $c = 2, \ldots, C$.

We also define the deterministic return on the risk-free asset denominated in currency *c* as $X_{c0} = x_{c0}$.

3.1.4 We define the following parameters, where, as above, for $c = 1, \ldots, C$, $i = 0$ denotes the risk-free asset denominated in that currency and $i = 1, \ldots, n_c$ denotes the risky assets issued in that currency:

— μ_{ci} is the expected return in currency *c* on risky asset *i* issued in that currency; i.e.: $\mu_{ci} = E\{X_{ci}\}$;
— $\sigma_{ci,dj}$ is the covariance of the return in currency *c* on risky asset *i* issued in that currency with the return in currency *d* on risky asset *j* issued in that currency; i.e.: $\sigma_{ci,dj} = \begin{cases} \mathrm{var}\{X_{ci}\} & \text{for } d = c, j = i; \\ \mathrm{cov}\{X_{ci}, X_{dj}\} & \text{otherwise;} \end{cases}$
— μ_c is the expected increase in the exchange rate of currency *c* per unit of currency 1; i.e.: $\mu_c = E\{X_c\}$;
— $\sigma_{c,di}$ is the covariance of the increase in the exchange rate of currency *c* per unit of currency 1 with the return in currency *d* on risky asset *i* issued in that currency; i.e.: $\sigma_{c,di} = \mathrm{cov}\{X_c, X_{di}\}$;
— $\sigma_{c,c}$ is the variance of the increase in the exchange rate of currency *c* per unit of currency 1; i.e.: $\sigma_{c,c} = \mathrm{var}\{X_c\}$.

Because investors have homogeneous expectations (assumption (2) of ¶3.1.2), the means, variances and covariances defined above are the same for all investors, regardless of the currency in which they measure their returns.

3.1.5 In the case of currency 1 the increase in the exchange rate per unit of currency 1 is trivially zero. For that currency we therefore have:

$$\mu_1 = 0; \tag{4}$$
$$\sigma_{1,di} = 0; \text{ and} \tag{5}$$
$$\sigma_{1,c} = 0. \tag{6}$$

Also, for the risk-free asset denominated in currency *c*, the return is deterministic, so we have:

$$\sigma_{c0,dj} = 0. \tag{7}$$

3.1.6 The variables defined in ¶¶3.1.3–3.1.5 relate to returns in a particular currency as measured in that currency and to exchange rates between that currency and currency 1. Now we need to consider the returns to investors who measure their returns in other currencies, for example a currency-*c* investor. For this purpose we define the following:

— X_{di}^{c} is the return in currency *c* on asset *i* issued in currency *d* for $c, d = 1, \ldots, C;\ i = 0, 1, \ldots, n_d$; i.e.:

$$X_{di}^{c} = X_{di} + X_d - X_c. \tag{8}$$

Because we are working with forces of increase in exchange rates, the increases are additive. We may then determine the following:

— μ_{di}^{c} is the expected return in currency *c* on asset *i* issued in currency *d* for $c, d = 1, \ldots, C;\ i = 0, 1, \ldots, n_d$; i.e.:

$$\mu_{di}^{c} = E\left\{X_{di} + X_d - X_c\right\} = \mu_{di} + \mu_d - \mu_c. \tag{9}$$

— $\sigma_{di,ej}^{c}$ is the covariance of the return in currency *c* on asset *i* issued in currency *d* with the return in currency *c* on risky asset *j* issued in currency *e*; i.e.:

$$\begin{aligned}
\sigma_{di,ej}^{c} &= \operatorname{cov}\left\{X_{di} + X_d - X_c, X_{ej} + X_e - X_c\right\} \\
&= \operatorname{cov}\left\{X_{di,} X_{ej}\right\} + \operatorname{cov}\left\{X_{di}, X_e\right\} - \operatorname{cov}\left\{X_{di}, X_c\right\} \\
&\quad + \operatorname{cov}\left\{X_d X_{ej}\right\} + \operatorname{cov}\left\{X_d, X_e\right\} - \operatorname{cov}\left\{X_d X_c\right\} \\
&\quad - \operatorname{cov}\left\{X_c X_{ej}\right\} - \operatorname{cov}\left\{X_c X_e\right\} + \operatorname{var}\left\{X_c\right\} \\
&= \sigma_{di,ej} + \sigma_{e,di} - \sigma_{c,di} + \sigma_{d,ej} + \sigma_{d,e} - \sigma_{c,d} - \sigma_{c,ej} - \sigma_{c,e} + \sigma_{c,c}.
\end{aligned} \tag{10}$$

3.1.7 Let p_{di}^{c} denote the value in currency c of investments in asset i issued in currency d held by currency-c investors, per unit of the total value in that currency of the assets held by such investors. The value of p_{di}^{c} is unknown; it is estimated through an optimisation process explained below. We now define the portfolio of risky assets held in currency d by a currency-c investor as:

$$\boldsymbol{p}_d^c = \begin{cases} \begin{pmatrix} p_{d0}^c \\ \vdots \\ p_{dn_d}^c \end{pmatrix} & \text{for } d \neq c; \\ \begin{pmatrix} p_{c1}^c \\ \vdots \\ p_{cn_c}^c \end{pmatrix} & \text{for } d = c. \end{cases} \tag{11}$$

This vector has $n_d + 1$ components for $d \neq c$ or n_c for $d = c$. This is because, for currency $d \neq c$, the risk-free asset denominated in currency d is included (as p_{d0}^c) as a risky asset in this portfolio, whereas for currency $d = c$, the risk-free asset denominated in that currency is not included, as it is not a risky asset. Then we define the market portfolio of a currency-c investor by combining the vectors $\boldsymbol{p}_1^c, \ldots, \boldsymbol{p}_C^c$ of equation (11) into the vector:

$$\boldsymbol{p}^c = \begin{pmatrix} \boldsymbol{p}_1^c \\ \vdots \\ \boldsymbol{p}_c^c \\ \vdots \\ \boldsymbol{p}_C^c \end{pmatrix}. \tag{12}$$

This vector has h unknown components, where h is the number of risky assets in which an investor can invest, viz.:

$$h = \sum_{c=1}^{C} n_c + C - 1. \tag{13}$$

We may express $\boldsymbol{p}^c$ as a simple vector of h components renumbered consecutively, viz.:

$$\boldsymbol{p}^c = \begin{pmatrix} p_1^c \\ \vdots \\ p_h^c \end{pmatrix}. \tag{14}$$

By definition, the elements of the vector $\boldsymbol{p}^c$ sum to 1; i.e.:

$$\sum_{\tau=1}^{h} p_{\tau}^{c} = 1. \tag{15}$$

3.1.8 Similarly, we define the return on the risky assets held in currency d by a currency-c investor as:

$$\boldsymbol{X}_{d}^{c} = \begin{cases} \begin{pmatrix} X_{d0}^{c} \\ \vdots \\ X_{dn_d}^{c} \end{pmatrix} & \text{for } d \neq c; \\ \begin{pmatrix} X_{c1}^{c} \\ \vdots \\ X_{cn_c}^{c} \end{pmatrix} & \text{for } d = c; \end{cases} \tag{16}$$

where X_{di}^{c} is the return on risky asset i issued in currency d measured in currency c (equation (8)). Then we define the return on the portfolio of risky assets held by a currency-c investor as the h-component vector:

$$\boldsymbol{X}^{c} = \begin{pmatrix} X_{1}^{c} \\ \vdots \\ X_{h}^{c} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_{1}^{c} \\ \vdots \\ \boldsymbol{X}_{c}^{c} \\ \vdots \\ \boldsymbol{X}_{C}^{c} \end{pmatrix}. \tag{17}$$

3.1.9 We similarly define the expected return on the risky assets held in currency d by a currency-c investor as:

$$\boldsymbol{\mu}_{d}^{c} = \begin{cases} \begin{pmatrix} \mu_{d0}^{c} \\ \vdots \\ \mu_{dn_d}^{c} \end{pmatrix} & \text{for } d \neq c; \\ \begin{pmatrix} \mu_{c1}^{c} \\ \vdots \\ \mu_{cn_c}^{c} \end{pmatrix} & \text{for } d = c; \end{cases} \tag{18}$$

where μ_{di}^{c} is the expected return on risky asset i issued in currency d measured in currency c (equation (9)). Then we define the expected return on the portfolio of risky assets held by a currency-c investor as the h-component vector:

$$\boldsymbol{\mu}^c = \begin{pmatrix} \mu_1^c \\ \vdots \\ \mu_h^c \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^c \\ \vdots \\ \boldsymbol{\mu}_c^c \\ \vdots \\ \boldsymbol{\mu}_C^c \end{pmatrix}. \tag{19}$$

3.1.10 Also, we define the covariance matrices of the returns on the risky assets held in currency d with those on the risky assets held in currency e by a currency-c investor as:

$$\boldsymbol{\Sigma}_{de}^c = \begin{cases} \begin{pmatrix} \sigma_{d0,e0}^c & \cdots & \sigma_{d0,en_e}^c \\ \vdots & \ddots & \vdots \\ \sigma_{dn_d,e0}^c & \cdots & \sigma_{dn_d,en_e}^c \end{pmatrix} & \text{for } d \neq c, e \neq c; \\ \begin{pmatrix} \sigma_{c1,e0}^c & \cdots & \sigma_{c1,en_1}^c \\ \vdots & \ddots & \vdots \\ \sigma_{cn_c,e0}^c & \cdots & \sigma_{cn_c,en_e}^c \end{pmatrix} & \text{for } d = c, e \neq c; \\ \begin{pmatrix} \sigma_{d0,c1}^c & \cdots & \sigma_{d0,cn_c}^c \\ \vdots & \ddots & \vdots \\ \sigma_{dn_d,c1}^c & \cdots & \sigma_{dn_d,cn_c}^c \end{pmatrix} & \text{for } d \neq c, e = c; \\ \begin{pmatrix} \sigma_{c1,c1}^c & \cdots & \sigma_{c1,cn_c}^c \\ \vdots & \ddots & \vdots \\ \sigma_{cn_c,c1}^c & \cdots & \sigma_{cn_c,cn_c}^c \end{pmatrix} & \text{for } d = c, e = c; \end{cases} \tag{20}$$

where $\sigma_{di,ej}^c$ is the covariance of the returns on risky assets i and j held in currencies d and e respectively by a currency-c investor (equation (10)). Then we define the $h \times h$ covariance matrix of the return on the portfolio of risky assets held by a currency-c investor as:

$$\boldsymbol{\Sigma}^c = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1h} \\ \vdots & \ddots & \vdots \\ \sigma_{h1} & \cdots & \sigma_{hh} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{11}^c & \cdots & \boldsymbol{\Sigma}_{1e}^c & \cdots & \boldsymbol{\Sigma}_{1C}^c \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_{d1}^c & \cdots & \boldsymbol{\Sigma}_{de}^c & \cdots & \boldsymbol{\Sigma}_{dC}^c \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_{C1}^c & \cdots & \boldsymbol{\Sigma}_{Ce}^c & \cdots & \boldsymbol{\Sigma}_{CC}^c \end{pmatrix}. \tag{21}$$

3.1.11 Now, from the definitions in equations (14) and (17), we may express the return on the portfolio of risky assets held by a currency-*c* investor as:

$$X_{\mathrm{M}}^{c} = \boldsymbol{p}^{c\prime}\boldsymbol{X}^{c}\,. \tag{22}$$

(We use the subscript M to denote that portfolio.) Similarly, from the definitions in equations (14) and (19), we may express the expected return on the portfolio of risky assets held by a currency-*c* investor as:

$$\mu_{\mathrm{M}}^{c} = E\left\{X_{\mathrm{M}}^{c}\right\} = \boldsymbol{p}^{c\prime}\boldsymbol{\mu}^{c}\,. \tag{23}$$

Also, from equations (14) and (21), we may express the variance of the return on the portfolio of risky assets held by a currency-*c* investor as:

$$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\} = \boldsymbol{p}^{c\prime}\boldsymbol{\Sigma}^{c}\boldsymbol{p}^{c}\,. \tag{24}$$

3.1.12 In terms of the CAPM, currency-*c* investors determine their portfolio of risky assets in quarter *t* by maximising

$$k_t = \frac{\hat{\mu}_{\mathrm{M},t}^{c} - z_{c0,t}}{\sqrt{\hat{\sigma}_{\mathrm{M,M},t}^{c}}}; \tag{25}$$

where:

$\hat{\mu}_{\mathrm{M},t}^{c}$ is the *ex-ante* estimate of the expected return to a currency-*c* investor on her/his portfolio during quarter *t*;

$z_{c0,t}$ is the observed risk-free rate in currency *c* during quarter *t*; and

$\hat{\sigma}_{\mathrm{M,M},t}^{c}$ is the *ex-ante* estimate of the variance of the return to a currency-*c* investor on her/his portfolio during quarter *t*.

3.1.13 For the purposes of this paper we assume that, during each period considered, $\hat{\sigma}_{\mathrm{M,M},t}^{c}$ is constant quarter by quarter, so that we may denote it $\hat{\sigma}_{\mathrm{M,M}}^{c}$. It would not be appropriate to assume that $\hat{\mu}_{\mathrm{M},t}^{c}$ is constant, since this would allow a negative market risk premium $\hat{\mu}_{\mathrm{M}}^{c} - z_{c0,t}$ when $z_{c0,t}$ is large. Instead we assume that the market risk premium

$$\pi^{c} = \hat{\mu}_{\mathrm{M},t}^{c} - z_{c0,t}$$

is constant, by setting:

$$\pi^{c} = \frac{1}{T}\sum_{t=1}^{T}\left(z_{\mathrm{M},t}^{c} - z_{c0,t}\right);$$

where $z_{\mathrm{M},t}^{c}$ is the observed return on a currency-*c* investor's market portfolio during quarter *t*. Equation (25) may then be restated as:

$$k = \frac{\hat{\mu}^c_{\mathrm{M}} - \hat{\mu}^c_{c0}}{\sqrt{\hat{\sigma}^c_{\mathrm{M,M}}}}; \tag{26}$$

where:

$\hat{\mu}^c_{\mathrm{M}}$ is the sample mean of the expected return to a currency-*c* investor on her/his portfolio during the period; i.e.

$$\hat{\mu}^c_M = \frac{1}{T}\sum_{t=1}^{T} z^c_{\mathrm{M},t};$$

$\hat{\mu}^c_{c0}$ is the sample mean of the risk-free rate in currency *c* during the period; i.e.:

$$\hat{\mu}^c_{c0} = \frac{1}{T}\sum_{t=1}^{T} z_{c0,t} \text{ ; and}$$

as defined above, $\hat{\sigma}^c_{\mathrm{M,M}}$ is the constant *ex-ante* estimate of the variance of the return to a currency-*c* investor on her/his portfolio during quarter *t*.

3.2 Derivation of the Necessary and Sufficient Condition

3.2.1 As shown, e.g., by Elton & Gruber (1995: 98–100), the value of $\boldsymbol{p}^c$ that maximises *k* is:

$$\boldsymbol{p}^c = \frac{1}{\boldsymbol{w}^{c\prime}\mathbf{1}}\boldsymbol{w}^c; \tag{27}$$

where·

$$\boldsymbol{w}^c = \left(\hat{\boldsymbol{\Sigma}}^c\right)^{-1}\left(\hat{\boldsymbol{\mu}}^c - \hat{\mu}^c_{c0}\mathbf{1}\right); \tag{28}$$

$$\hat{\boldsymbol{\mu}}^c = \begin{pmatrix} \hat{\boldsymbol{\mu}}^c_1 \\ \vdots \\ \hat{\boldsymbol{\mu}}^c_c \\ \vdots \\ \hat{\boldsymbol{\mu}}^c_C \end{pmatrix}; \tag{29}$$

$$\hat{\boldsymbol{\mu}}^c_d = \begin{cases} \begin{pmatrix} \hat{\mu}^c_{d0} \\ \vdots \\ \hat{\mu}^c_{dn_d} \end{pmatrix} & \text{for } d \neq c; \\ \begin{pmatrix} \hat{\mu}^c_{c1} \\ \vdots \\ \hat{\mu}^c_{cn_c} \end{pmatrix} & \text{for } d = c; \end{cases} \tag{30}$$

$$\hat{\mu}_{di}^{c} = \frac{1}{T}\sum_{t=1}^{T} x_{di}^{c}; \tag{31}$$

and **1** is the h-component unit vector.

3.2.2 Since the CAPM applies for investors in each currency (assumption (1) of ¶3.1.2), it follows that, for asset i issued in currency d, the expected returns in currencies c and 1 are respectively:

$$\mu_{di}^{c} = x_{c0} + \frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right); \text{ and} \tag{32}$$

$$\mu_{di}^{1} = x_{10} + \frac{\sigma_{di,\mathrm{M}}^{1}}{\sigma_{\mathrm{M,M}}^{1}}\left(\mu_{\mathrm{M}}^{1} - x_{10}\right); \tag{33}$$

where $\sigma_{di,\mathrm{M}}^{c}$ is the covariance of the return in currency c on asset i issued in currency d with the return in currency c on the market portfolio of a currency-c investor, i.e.:

$$\sigma_{di,\mathrm{M}}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{M}}^{c}\right\} = \sum_{e=1}^{C}\sum_{j=0}^{n_e} p_{ej}^{c}\sigma_{di,ej}^{c} = \left(\boldsymbol{p}^{c}\right)'\boldsymbol{\sigma}_{di}^{c}; \text{ and} \tag{34}$$

$$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}, X_{\mathrm{M}}^{c}\right\} = \sum_{d=1}^{C}\sum_{i=0}^{n_d}\sum_{e=1}^{C}\sum_{j=0}^{n_e} p_{di}^{c}p_{ej}^{c}\sigma_{di,ej}^{c} = \left(\boldsymbol{p}^{c}\right)'\boldsymbol{\sigma}_{M,c}^{c}; \tag{35}$$

where:

$$\boldsymbol{\sigma}_{di}^{c} = \begin{pmatrix} \sigma_{di,11}^{c} \\ \vdots \\ \sigma_{di,Cn_C}^{c} \end{pmatrix};$$

$$\boldsymbol{\sigma}_{\mathrm{M},c}^{c} = \begin{pmatrix} \sigma_{11,\mathrm{M}}^{c} \\ \vdots \\ \sigma_{Cn_C,\mathrm{M}}^{c} \end{pmatrix}; \text{ and}$$

$$p_{c0}^{c} = 0.$$

In the vectors $\boldsymbol{\sigma}_{di}^{c}$ and $\boldsymbol{\sigma}_{\mathrm{M},c}^{c}$ the components $\sigma_{di,c0}^{c}$ and $\sigma_{c0,\mathrm{M}}^{c}$ are omitted.

3.2.3 Consider the investment return in currency c on asset i issued in currency d. From equations (4) and (9) the expected return on this asset is:

$$\mu_{di}^{c} = \mu_{di}^{1} - \mu_{c}. \tag{36}$$

Now, substituting equations (32) and (33) into equation (36) we have:

$$x_{c0} + \frac{\sigma^c_{di,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - x_{c0}\right) = x_{10} + \frac{\sigma^1_{di,\mathrm{M}}}{\sigma^1_{\mathrm{M,M}}}\left(\mu^1_{\mathrm{M}} - x_{10}\right) - \mu_c. \tag{37}$$

Equation (37) holds if and only if:

$$\sigma^c_{di,\mathrm{M}} = \frac{\sigma^c_{\mathrm{M,M}}}{\mu^c_{\mathrm{M}} - x_{c0}}\left(\frac{\mu^1_{\mathrm{M}} - x_{10}}{\sigma^1_{\mathrm{M,M}}}\sigma^1_{di,\mathrm{M}} + x_{10} - x_{c0} - \mu_c\right). \tag{38}$$

3.2.4 Now from equations (4) and (8) the expected return to a currency-*c* investor on the risk-free asset in currency 1 is:

$$\mu^c_{10} = x_{10} - \mu_c; \tag{39}$$

which, under the CAPM, must be:

$$x_{10} - \mu_c = x_{c0} + \frac{\sigma^c_{10,\mathrm{M}}}{\sigma^c_{\mathrm{M,M}}}\left(\mu^c_{\mathrm{M}} - x_{c0}\right);$$

i.e.:

$$\frac{\sigma^c_{\mathrm{M,M}}}{\mu^c_{\mathrm{M}} - x_{c0}}\left(x_{10} - x_{c0} - \mu_c\right) = \sigma^c_{10,\mathrm{M}}. \tag{40}$$

3.2.5 On substituting equation (40) into equation (38) we have:

$$\sigma^c_{di,\mathrm{M}} = \frac{\sigma^c_{\mathrm{M,M}}\left(\mu^1_{\mathrm{M}} - x_{10}\right)}{\sigma^1_{\mathrm{M,M}}\left(\mu^c_{\mathrm{M}} - x_{c0}\right)}\sigma^1_{di,\mathrm{M}} + \sigma^c_{10,\mathrm{M}}. \tag{41}$$

Now

$$\begin{aligned}\sigma^c_{10,\mathrm{M}} &= \mathrm{cov}\left\{x_{10} - X_c, X^c_{\mathrm{M}}\right\}\\ &= -\mathrm{cov}\left\{X_c, X^c_{\mathrm{M}}\right\}\\ &= -\sigma^c_{c,\mathrm{M}};\end{aligned}$$

where:

$$\sigma^c_{c,\mathrm{M}} = \sum_{e=1}^{C}\sum_{j=0}^{n_e} p^c_{ej}\sigma_{c,ej} = \left(\boldsymbol{p}^c\right)'\boldsymbol{\sigma}_c; \text{ and} \tag{42}$$

$$\boldsymbol{\sigma}_c = \begin{pmatrix}\sigma_{c,11}\\ \vdots \\ \sigma_{c,Cn_c}\end{pmatrix}.$$

In the vector $\boldsymbol{\sigma}_c$ the component $\sigma_{c,c0}$ is omitted. Equation (38) may therefore be expressed as:

$$\sigma^{c}_{di,\mathrm{M}} + \sigma^{c}_{c,\mathrm{M}} = a_c \sigma^{1}_{di,\mathrm{M}}; \tag{43}$$

where:

$$a_c = \frac{\sigma^{c}_{\mathrm{M,M}} \left(\mu^{1}_{\mathrm{M}} - x_{10} \right)}{\sigma^{1}_{\mathrm{M,M}} \left(\mu^{c}_{\mathrm{M}} - x_{c0} \right)}. \tag{44}$$

Given equations (4), (8) and (9), it follows from the fact that equation (36) holds if and only if equation (37) holds, that equation (43) holds if and only if equation (32) holds. Equation (43) is thus both a sufficient condition and a necessary condition for the single-factor CAPM to apply in a multi-currency world. In the estimation of the variance and covariance parameters $\sigma_{ci,dj}$, $\sigma_{c,di}$ and $\sigma_{c,c}$, equation (43) must be used as a constraint.

3.2.6 In theory, the variance and covariance parameters $\sigma_{ci,dj}$, $\sigma_{c,di}$ and $\sigma_{c,c}$, may be found by constrained maximum likelihood estimation, that is by maximising the likelihood of the observed values subject to the constraint of equation (43). In practice this is difficult. Problems relating to this approach are explained below.

3.3 Formulation of the SFM-CAPM

3.3.1 Equation (43) may also be expressed as:

$$\left(\sigma^{c}_{di,\mathrm{M}} + \sigma^{c}_{c,\mathrm{M}} \right) \frac{\mu^{c}_{\mathrm{M}} - x_{c0}}{\sigma^{c}_{\mathrm{M,M}}} = \sigma^{1}_{di,\mathrm{M}} \frac{\mu^{1}_{\mathrm{M}} - x_{10}}{\sigma^{1}_{\mathrm{M,M}}}. \tag{45}$$

In other words, for all *c*:

$$\left(\sigma^{c}_{di,\mathrm{M}} + \sigma^{c}_{c,\mathrm{M}} \right) \frac{\mu^{c}_{\mathrm{M}} - x_{c0}}{\sigma^{c}_{\mathrm{M,M}}} = \kappa_{di}. \tag{46}$$

In the case of $c = 1$, from equations (5) and (42), $\sigma^{1}_{1,\mathrm{M}} = 0$. We may refer to κ_{di} as the risk premium for asset *i* in currency *d* adjusted for exchange-rate risk. The value of κ_{di} will differ according to which currency is chosen as currency 1. However, the equality of the adjusted risk premia will be unaffected by that choice. Given that choice, the adjustment is the same for all (*d*, *i*).

3.3.2 Equation (46) applies *ex ante*. It will not necessarily apply *ex post*. Because equation (26) is expressed in terms of the *ex-post* mean $\hat{\mu}^{c}_{c0}$ of quarterly risk-free rates defined in ¶3.1.13 and not in terms of the risk-free rate x_{c0} for a particular quarter, which is known at the start of that quarter, the optimal portfolios, and the variables in equation (46) that arise from that optimisation, will not generally result in the equality shown in that equation.

3.3.3 The equality of the adjusted risk premium κ_{di} for all investors may be heuristically explained as comprising two factors: the normal beta for the currency-*c* investor's return on asset *i* in currency *d*, i.e.:

$$\beta_{di}^{c} = \frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}; \tag{47}$$

and an additional beta for that investor's exchange-rate risk relative to currency 1, i.e.:

$$\beta_{c}^{c} = \frac{\sigma_{c,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}. \tag{48}$$

The use of exchange-rate risk relative to currency 1 instead of currency *d* is counter-intuitive. Although the value of beta may be analysed as comprising two factors, both factors relate to the price of covariance in currency *c*, so that in effect we are dealing with one factor. Equation (46) may therefore be expressed as:

$$\beta_{di,c}^{c}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right) = \kappa_{di}; \qquad 49)$$

where:

$$\beta_{di,c}^{c} = \beta_{di}^{c} + \beta_{c}^{c}. \tag{50}$$

$\beta_{di,c}^{c}$ specifies the single factor. The SFM-CAPM may therefore be expressed as:

$$E\left\{X_{di}^{c}\right\} = x_{d0}^{c} + \beta_{di,c}^{c}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right); \tag{51}$$

i.e.:

$$\mu_{di}^{c} = x_{d0}^{c} + \kappa_{di}; \tag{52}$$

where:

$$\kappa_{di} = \beta_{di,c}^{c}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right) = \sigma_{di,\mathrm{M}}^{1}\frac{\mu_{\mathrm{M}}^{1} - x_{10}}{\sigma_{\mathrm{M,M}}^{1}}. \tag{53}$$

3.3.4 If strict purchasing-power parity holds then the SFM-CAPM reduces to the GCAPM. In this case, in real terms (or if there is no inflation):

$$\sigma_{c,\mathrm{M}}^{c} = 0;$$

and, for all currencies *c* and *e*:

$$\beta_{di}^{c} = \beta_{di}^{e}.$$

Equation (51) then reduces to equation (2). As explained in ¶2.3.2, equation (2) then applies regardless of the currency in which the investor measures returns.

3.4 Alternative Approaches

3.4.1 As explained in ¶3.2.6, whilst the approach outlined in section 3.2 is theoretically appealing, it is problematic to apply in practice. The method is complicated; it requires a lengthy program, and it necessitates an iterative process, which may fail to converge. Furthermore, it involves the inversion of a matrix that would run to over 200 rows and columns for the applications envisaged in this paper. Problems with singular matrices arise, especially for the small number of observations typically available at quarterly intervals in each period during which means, variances and covariances may be reasonably assumed to be constant.

3.4.2 Another problem is that, because of the linear constraint in equation (43), there is no reason why a constrained variance is necessarily positive. For a large sample variance it is unlikely that the constrained estimate of the variance will be negative, but for a small sample variance the constrained estimate may turn out to be negative. Similar problems may arise with correlation coefficients outside of the range [–1, 1].

3.4.3 For the reasons outlined above, alternative approaches are considered.

3.4.4 It might be argued that equation (45) could be used to estimate μ_{M}^{c}; i.e.:

$$\mu_{\mathrm{M}}^{c} = x_{c0} + \frac{\sigma_{di,\mathrm{M}}^{1}\sigma_{\mathrm{M,M}}^{c}}{\left(\sigma_{di,\mathrm{M}}^{c} + \sigma_{c,\mathrm{M}}^{c}\right)\sigma_{\mathrm{M,M}}^{1}}\left(\mu_{\mathrm{M}}^{1} - x_{10}\right). \tag{54}$$

The problem with this argument is that it presupposes that, whilst μ_{M}^{1} is known, μ_{M}^{c} is unknown for $c = 2,\ldots,C$. If that is accepted as true then it would not be inappropriate to use equation (54), but the result would not be independent of the choice of currency 1.

3.4.5 Another approach is to use equation (49) to calculate an observed value of κ_{di} for each currency, using ordinary least squares based on the usual sample means, variances and covariances. This is dealt with in section 3.5 below.

3.5 Least-Squares Estimation of the Betas

3.5.1 Let $\hat{\kappa}_{di}^{c}$ denote the observed value of κ_{di} for a currency-*c* investor. Equation (49) states that:

$$\beta_{di,c}^{c}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right) = \kappa_{di}.$$

In order to estimate κ_{di} using least squares, we require:

$$\frac{\partial}{\partial \kappa_{di}}\left[\sum_{c=1}^{C}\left\{\beta_{di,c}^{c}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right) - \kappa_{di}\right\}^{2}\right] = 0;$$

i.e.:

$$-2\left[\sum_{c=1}^{C}\left\{\beta_{di,c}^{c}\left(\mu_{\mathrm{M}}^{c}-x_{c0}\right)-\kappa_{di}\right\}\right]=0;$$

and hence:

$$\begin{aligned}\hat{\kappa}_{di}&=\frac{1}{C}\sum_{c=1}^{C}\left\{\hat{\beta}_{di,c}^{c}\left(\hat{\mu}_{\mathrm{M}}^{c}-x_{c0}\right)\right\}\\&=\frac{1}{C}\sum_{c=1}^{C}\hat{\kappa}_{di}^{c};\end{aligned}\tag{55}$$

where:

$$\hat{\kappa}_{di}^{c}=\hat{\beta}_{di,c}^{c}\left(\hat{\mu}_{\mathrm{M}}^{c}-x_{c0}\right).$$

In equation (55) the summation is with respect to currencies for which asset i in currency d is in the opportunity set of risky assets. For $i=0$ this excludes $c=d$ as this is the risk-free asset in that currency. For $i=0$ the summation is therefore with respect to $C-1$ currencies and the sum is therefore divided by $C-1$ to give the estimate.

3.5.2 We may then estimate $\beta_{di,c}^{c}$ using equation (49) as:

$$\hat{\beta}_{di,c}^{c}=\frac{\hat{\kappa}_{di}}{\mu_{\mathrm{M}}^{c}-x_{c0}}.\tag{56}$$

This method is considerably simpler than that of constrained maximum-likelihood estimation. Estimates of κ_{di} and $\beta_{di,c}^{c}$ were therefore determined by this method.

4. APPLICATION

For illustrative purposes the method outlined in section 3.5 was applied to a selection of currencies. In section 4.1 an overview of the data is given. The results are presented in section 4.2.

4.1 Data

4.1.1 The currencies selected are:

$$c=\begin{cases}1\text{ for USA dollars;}\\2\text{ for UK pounds;}\\3\text{ for South African rands; and}\\4\text{ for Turkish lira;}\end{cases}\tag{57}$$

so that $C = 4$. The selection is influenced by the authors' interests and the data most easily available to them, but the application is intended to be illustrative; care has been taken to specify the data sources and calculations so as to facilitate consistent applications to other currencies.

4.1.2 Quarterly forces of return and of increases in exchange rates were used. The methods were applied both to real returns and to nominal returns. The real returns and increases in exchange rates were calculated as:

$$\tilde{x}_{ci}(t) = x_{ci}(t) - \theta_c(t); \text{ and} \tag{58}$$

$$\tilde{x}_c(t) = x_c(t) - \theta_c(t) \tag{59}$$

respectively; where:

$x_{ci}(t)$ is the return in currency *c* on asset *i* issued in that currency during quarter *t*;

$x_c(t)$ is the increase in the exchange rate of currency *c* per unit of currency 1 during that quarter; and

$\theta_c(t)$ is the force of inflation in currency *c* during quarter *t*.

Because we are working with forces, the relationships in equations (58) and (59) are linear.

4.1.3 The risky assets covered by the application comprised equities, and conventional and inflation-protected government bonds (index-linked bonds) of selected maturities. For nominal returns the risk-free asset was the conventional bond with a maturity of one quarter and for real returns it was the corresponding inflation-protected bond.

4.1.4 The period covered by the application was from 1975Q2 to 2012Q1. The calculation of $x_{ci}(t)$, $x_c(t)$ and $\theta_c(t)$ from the data available[1] entailed intensive work, some of which relied on assumptions and estimations by the authors. These calculations are dealt with in detail in a note entitled "How a single-factor CAPM works in a multi-currency world: information on the determination of data," which, together with a spreadsheet file showing the determination of the data required, is available free of charge from the authors. As explained in that note, data are not available for every series throughout the period and in some cases the data were not acceptable for the purposes of this paper. The quarters from which acceptable returns (or, in the case of inflation and exchange rates, acceptable forces of inflation and forces of increase in exchange rates respectively) could be calculated from the data available for the respective series are shown in Table 1.

4.1.5 The type of bond used is also shown in Table 1: 'coupon' denotes coupon-paying bonds and 'ZCBs' denotes zero-coupon bonds. These are not necessarily the type of

1 Sources: USA Federal Reserve Bank; Bureau of Labor Statistics, U.S. Department of Labor; Bank of England; INet; Central Bank of the Republic of Turkey; Republic of Turkey Prime Ministry, Undersecretariat of Treasury, Public Finance; Istanbul Stock Exchange

bond in issue; where possible, returns were determined for zero-coupon bonds in order to avoid inaccuracies relating to the amount of the coupon on different bonds used for the determination of yield-curve data available. For each series of bonds, two terms to redemption were chosen: the short term of one quarter and a long term depending on the data available. The column headed 'long term (qtrs.)' shows the term to redemption chosen as the long term (in quarters). In Table 1, 'SA' refers to South Africa and 'TR' to Turkey.

Table 1 Periods for which acceptable returns could be calculated

Currency		Series	Bonds		Available from
no.	country		type	long term (qtrs.)	
1	USA	conventional bonds	coupon	80	1975Q2
		index-linked bonds	ZCBs	40	2003Q2
		equities			1975Q2
		inflation rates			1975Q2
2	UK	conventional zero-coupon bonds	ZCBs	64	1975Q2
		index-linked bonds	ZCBs	64	1985Q2
		equities			1975Q2
		exchange rates			1975Q2
		inflation rates			1975Q2
3	SA	conventional bonds	coupon	80	1975Q2
		index-linked bonds	ZCBs	40	2005Q3
		equities			1975Q2
		exchange rates			1975Q2
		inflation rates			1975Q2
4	TR	conventional bonds	ZCBs	8	1985Q3
		index-linked bonds	ZCBs	40	2009Q4
		equities			1986Q1
		exchange rates			1975Q2
		inflation rates			1982Q2

4.1.6 In the light of the information in Table 1 it was decided to use various datasets for nominal returns and various datasets (not necessarily the same) for real returns. These datasets, comprising the periods, and the assets included in them, are shown in Table 2. In that table, 'e, cb' means equities and conventional bonds and 'ilb' means index-linked bonds.

4.1.7 Some of the periods in Table 2 are too short for the estimation of reliable variance–covariance matrices; they are included for the sake of inclusivity and to indicate how they may affect the results. On the other hand, it must be recognised that means and covariances may change over time, so the use of excessively long periods is inappropriate. However, long periods have been included for the purposes of illustration.

Table 2 Periods used

Dataset	Period		USA		UK		SA		TR	
			e, cb	ilb	e, cb	ilb	e, cb	ilb	e, cb	ilb
nominal returns										
1	1975Q2	1985Q4	√		√		√			
2	1986Q1	1995Q4	√		√		√		√	
3	1996Q1	2005Q2	√		√		√		√	
4	2005Q3	2012Q1	√		√		√		√	
5	2005Q3	2012Q1	√	√	√	√	√	√		
6	2009Q4	2012Q1	√	√					√	√
7	1975Q2	2012Q1	√		√		√			
8	1986Q1	2012Q1	√		√		√		√	
real returns										
1	2003Q2	2009Q3	√	√	√	√				
2	2005Q3	2009Q3	√	√	√	√	√	√		
3	2009Q4	2012Q1	√	√					√	√
4	2005Q3	2012Q3	√	√	√	√	√	√		
5	2003Q2	2012Q3	√	√	√	√				

4.2 Results

4.2.1 For nominal and real returns, and for each period listed in Table 2, values of beta were calculated as follows:

— the unbiased sample betas based on equation (47); and
— the least-squares estimates of the betas based on equation (56).

The results of these calculations are set out in this section.

4.2.2 It was expected that the least-squares estimates would be approximately equal to the sample betas, with some shift to represent the effects of the constraint and some

noise. An upward shift would indicate that the sample betas are understated and a downward shift would indicate that they are overstated. The shifts indicate the corrections required to the sample betas by the underlying assumptions of the SFM-CAPM.

4.2.3 Figure 1 shows the relationship of the least-squares estimates of the betas to the sample betas for nominal returns for the datasets listed in Table 2. In that figure, every dot shows the values of these betas for a particular asset in a particular currency during a particular period.

4.2.4 Apart from dataset 4, the least-squares betas correspond to the expectations stated in ¶4.2.2. They are clustered about a straight line passing through the vertical axis below the horizontal axis, with a slope less than 1. In general, therefore, the least-squares betas are less than the sample betas; on the assumptions of the SFM-CAPM, sample betas tend to be overstated. There is considerable noise in the shift from sample betas to least-squares betas.

4.2.5 As shown in Table 2, dataset 4 is for returns on conventional bonds and equities in all four currencies during the period 2005Q3 to 2012Q1. Figure 2 shows a comparison for this dataset. Here the results for different investor currencies have been identified.

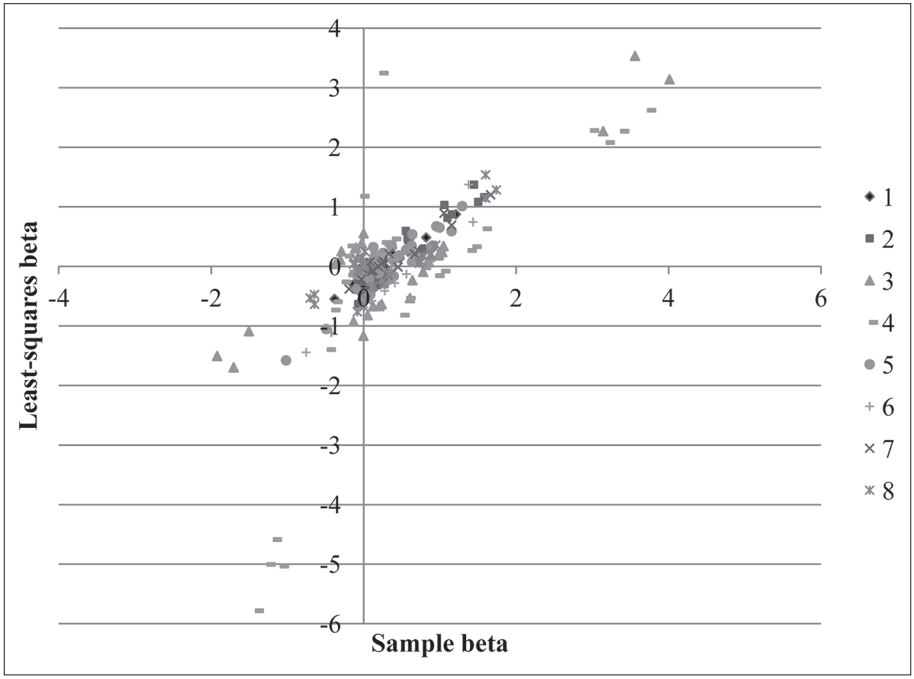

Figure 1 Least-squares beta versus sample beta: nominal returns

4.2.6 It is clear from Figure 2 that, for dataset 4, there are substantial differences between the betas—both the sample betas and the least-squares estimates—for different investor currencies. For USA-dollar investors the sample betas are relatively high and there is a downward shift in the least-squares estimates. For UK-pound investors the sample betas are relatively low and the least-squares estimates are very sensitive. For SA-rand and TR-lira investors all the betas are virtually zero. It appears that these effects are produced by substantial long and short positions in certain assets, giving market portfolios for USA-dollar and UK-pound investors that are strongly correlated (positively or negatively) with those assets.

4.2.7 Figure 3 shows the relationship of the least-squares betas to the sample betas for real returns for the datasets listed in Table 2.

4.2.8 Figure 3 shows that the betas of real returns behaved considerably better than those of nominal returns in that there is considerably less noise and no anomalous dataset. Again there is evidence that, on the assumptions of the SFM-CAPM, sample betas tend to be overstated. Here the datasets are more clearly differentiated as broken lines. Each line more or less defines a particular currency in which investors measure returns. This phenomenon is caused by the fact that, where $\hat{\kappa}_{di}^{c} < \hat{\kappa}_{di}$, $\hat{\beta}_{di,c}^{c}$ is shifted

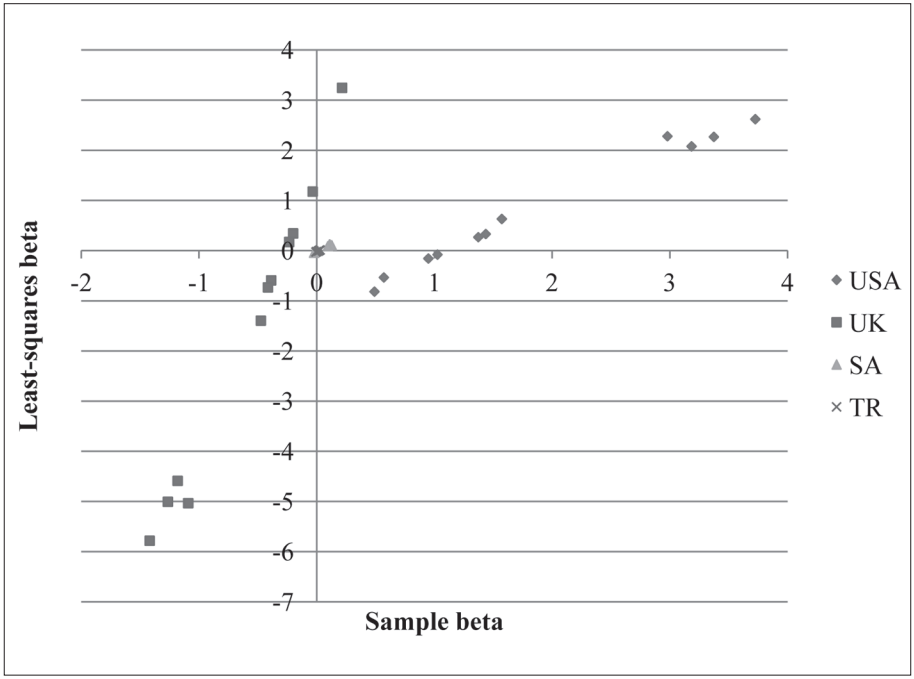

Figure 2 Least-squares beta versus sample beta: nominal returns: dataset 4

upwards and vice versa (equation (56)). In fact the separation of the lines represents the *ex-post* departures from the theoretical constraint, viz.:

$$\hat{\kappa}_{di}^{c} = \hat{\kappa}_{di} \text{ for all } c.$$

4.2.9 The relatively good behaviour of the SFM-CAPM for real returns appears to indicate that, in real terms, increases in exchange rates are relatively small, so that purchasing power is close to parity. This means that, as explained in ¶¶2.3.2 and 3.3.4, the SFM-CAPM is closer to the GCAPM in real terms than in nominal terms.

5. CONCLUSIONS

5.1 Summary

5.1.1 It is shown above that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the *ex-ante* variances and covariances of returns. The resulting SFM-CAPM model developed in this paper may be specified as:

$$E\left\{X_{di}^{c}\right\} = x_{c0} + \beta_{di,c}^{c}\left(E\left\{X_{\mathrm{M}}^{c}\right\} - x_{c0}\right);$$

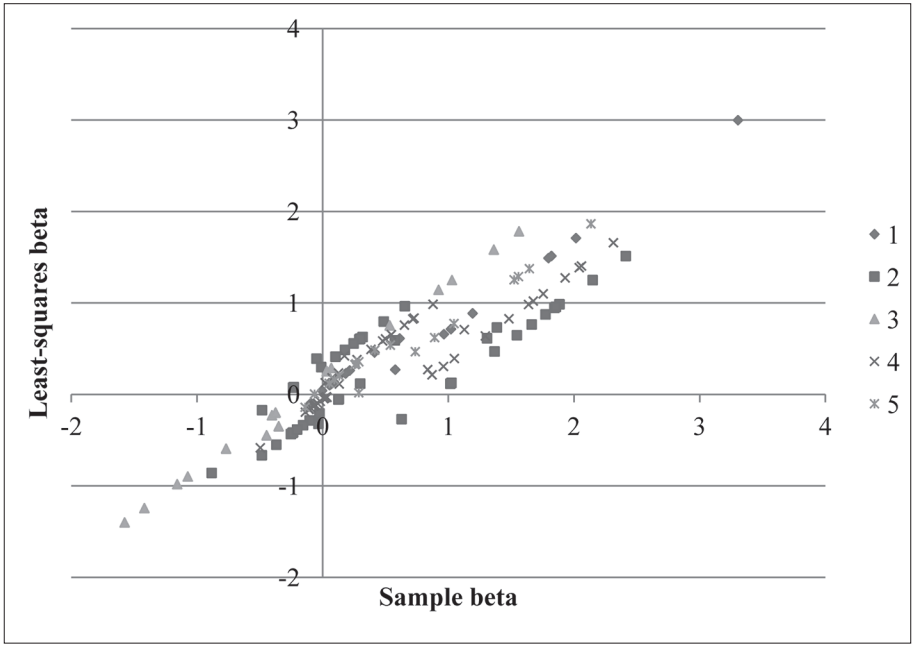

Figure 3 Least-squares beta versus sample beta: real returns

where:

X_{di}^{c} is the return in currency *c* on asset *i* issued in currency *d* for $c, d = 1, \ldots, C;\ i = 0, 1, \ldots, n_d$;

x_{c0} is the return on the risk-free asset denominated in currency *c*;

X_{M}^{c} is the return in currency *c* on the optimal market portfolio of a currency-*c* investor;

$$\beta_{di,c}^{c} = \beta_{di}^{c} + \beta_{c}^{c};$$

$$\beta_{di}^{c} = \frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}};$$

$$\beta_{c}^{c} = \frac{\sigma_{c,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}};$$

$$\sigma_{di,\mathrm{M}}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{M}}^{c}\right\};$$

$$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\};$$

$$\sigma_{c,\mathrm{M}}^{c} = \mathrm{cov}\left\{X_{c}, X_{\mathrm{M}}^{c}\right\};$$

X_{c} is the increase in the exchange rate of currency *c* relative to an arbitrarily chosen currency 1.

The constrained may be expressed as:

$$\sigma_{di,\mathrm{M}}^{c} + \sigma_{c,\mathrm{M}}^{c} = a_{c}\sigma_{di,\mathrm{M}}^{1};$$

where:

$$a_{c} = \frac{\sigma_{\mathrm{M,M}}^{c}\left(\mu_{\mathrm{M}}^{1} - x_{10}\right)}{\sigma_{\mathrm{M,M}}^{1}\left(\mu_{\mathrm{M}}^{c} - x_{c0}\right)};$$

or alternatively, for all *c*:

$$\left(\sigma_{di,\mathrm{M}}^{c} + \sigma_{c,\mathrm{M}}^{c}\right)\frac{\mu_{\mathrm{M}}^{c} - x_{c0}}{\sigma_{\mathrm{M,M}}^{c}} = \kappa_{di}.$$

5.1.2 It is found that the application of this constraint in the maximum-likelihood estimation of the variance–covariance matrix of returns is difficult. The difficulties with that approach are explained and an alternative approach, using ordinary least squares, is developed.

5.1.3 The least-squares method is applied to the USA dollar, the UK pound, the South African rand and the Turkish lira. The datasets used cover a selection of periods ranging from 1975 to 2012. The assets considered, to the extent that acceptable data were available, included, for each currency, two conventional bonds, two index-linked bonds and a comprehensive index of equities. The two bonds were a short (one-quarter) bond and a long-term bond. For nominal returns the short conventional bond was taken as the risk-free asset and for real returns the short index-linked bond was taken as such.

5.1.4 It was found that, for nominal returns, most datasets gave reasonable results. In general, the least-squares betas are less than the sample betas; on the assumptions of the SFM-CAPM, *ex-post* sample betas tend to be overstated. There was considerable noise in the shift from sample betas to least-squares betas. One dataset gave results that could not be considered reasonable. This was apparently due to substantial short positions in the optimal portfolios. This suggests that the constraints should be applied against short positions.

5.1.5 The results for real returns were better than those for nominal returns. This is apparently due to the approximate parity of purchasing power. Again the evidence is that sample betas tend to be overstated. Mean–variance analysis in general, and the CAPM in particular, must relate to consumption and therefore to purchasing power. In principle, it would therefore be better to develop stochastic models for actuarial use in real terms, particularly if the liabilities being modelled are effectively so expressed, as in defined-benefit pension funds. The principal problem with the use of real returns is that the history of index-linked bonds is shorter. Also, there is little trade and no primary market at the short end of the real yield curve, making it difficult to estimate real risk-free returns.

5.2 Further Research

5.2.1 In practice the stochastic modelling of the assets and liabilities of a long-term financial institution requires the use of time series in which the expected returns on assets and the expected forces of inflation, and perhaps their variances and covariances, may vary over time. This means that the application of the SFM-CAPM to such modelling will necessitate the use of the method in a time series. The variances and covariances, or the parameters governing autoregressive conditional heteroskedasticity (ARCH) effects, subject to the constraint required for the use of the SFM-CAPM, will need to be estimated in that context. That is the subject of further research. Such research should demonstrate the development of equilibrium models for applications that are specific both to a particular country and to a particular type of financial institution, with allowance for investment in other currencies. Because the SFM-CAPM is an equilibrium model, it is well suited for such applications. The advantages of equilibrium models for such purposes is that they do not assume that

the investor can outperform the market on a risk-adjusted basis, thus allowing market consistency, and that, unlike more general no-arbitrage models, they do not assume complete markets, thus allowing for the fact that the liabilities of a financial institution cannot be replicated in the market.

5.2.2 As Wilkie (op. cit.) points out, the effects of different ways of measuring returns and of different ways of defining the risk-free rate are mathematically analogous to the effect of different currencies in which investors measure returns. Thus, for example, instead of distinguishing investors by the currencies in which they measure returns, we could distinguish them by the mean term of their liabilities (and therefore the term that defines their risk-free rate) or by the extent to which their liabilities are expressed in real or nominal terms. On the other hand, for such applications there is no divergence from purchasing-power parity. The application of the principles explored in this paper to the use of the CAPM for a domestic market with different time horizons and different definitions of the risk-free asset requires further research.

5.2.3 The testing of hypotheses regarding the SFM-CAPM is another field of research. For the purposes of such testing, however, it must be recognised that our tests should, in principle, be based on *ex-ante* expectations, not necessarily *ex-post* estimations of those expectations. The use of the rational-expectations hypothesis to argue that the latter is an unbiased estimate of the former is at best only asymptotically true.

5.2.4 In the first place it may be best to estimate investors' *ex-ante* expected returns in the various asset classes instead of using *ex-post* estimations. This could be done by using actual market capitalisations to derive *ex-ante* expected returns instead of deriving optimal portfolios from *ex-post* sample means.

5.2.5 In the determination of optimal portfolios in this paper, no constraint was imposed on short positions. In the event, some large short positions were produced. Whilst there is no reason why modest short positions should not be accommodated, it would be necessary in practical applications to constrain short positions. This could be done either by using constrained optimisation or by using actual market capitalisations. This would eliminate the sort of anomaly found in nominal dataset 4.

5.2.6 As far as variances and covariances are concerned, a further difficulty with the testing of the SFM-CAPM will be that, by virtue of the constraint, the estimates of the variances and covariances will be biased. In essence, the constrained estimates are estimates of *ex-ante* variances and covariances, not *ex-post* estimates. This means that there will be no basis for the rejection of hypotheses relating to these estimates. The SFM-CAPM therefore belongs to the world of normative modelling: if the financial institution accepts the assumptions on which the model is based then it should accept the model derived from those assumptions. In the short run there is no basis for

asserting that the model is descriptively valid. But if and as it becomes accepted, it will become descriptively valid. In the mean time, if it is found that the constrained estimates are significantly different from the unconstrained estimates then there are three possibilities, viz. that:

— the rational expectations hypothesis is true and the SFM-CAPM is false; or
— the rational expectations hypothesis is false; or
— there is a type-1 error.

5.2.7 Repeated tests may reduce the probability of a type-1 error, but they will still not reduce the probability that the rational expectations hypothesis is false. The suggestion that normatively valid behaviour will become descriptively valid if decision-makers subscribe to it is an alternative to the rational expectations hypothesis. This tendency will be reinforced by the fear that, if one does not follow normative rationality then it will be more difficult to explain one's errors than if one does.

5.2.8 Instead of eliminating bias in *ex-ante* assumptions through the processing of *ex-post* information, here the decision-maker eliminates bias by adopting a rational basis for the determination of *ex-ante* assumptions, namely the constrained maximum-likelihood estimate proposed in this paper, even though those assumptions may be biased *ex post* (or, to challenge the dominant paradigm, even though the *ex-post* estimates are biased *ex ante*).

5.2.9 The complexities of the constrained maximum-likelihood method and the production of negative estimates of the variances of the returns on certain assets is not acceptable for practical application, nor is the production of correlation coefficients outside the range [–1, 1]. Further research is necessary to find ways of overcoming the problems associated with the constrained maximum-likelihood method. In the mean time, however, the least-squares estimation method may be used. Thanks to the relative simplicity of the latter method, it may well prove more popular with practitioners, but that does not obviate the need for further research on the constrained maximum-likelihood method.

5.2.10 On the other hand, if it is found that the constrained estimates are not significantly different from the unconstrained estimates then the SFM-CAPM could be applied without the constraint.

5.2.11 What is more likely is that the SFM-CAPM estimates will be significantly different from the unconstrained estimates for some currencies during some periods and not significantly different for other currencies during other periods. This will place the practitioner in a quandary: will her/his currencies and time horizon conform to the one case or the other? The quandary may be resolved by the suggestion offered here: that it is normative rationality that drives behaviour. If it fails to do so, at least

among long-term financial institutions advised by actuaries, then it is the profession that will have failed to convince its clients.

5.2.12 Whilst the principal interest of the authors is in the development of stochastic models for actuarial use, the SFM-CAPM clearly has wider application—for example in determining cost of capital. For such applications it is not necessarily envisaged that this model will replace other models, but subject to the results of the further research suggested here, there is no reason why the SFM-CAPM should not take its place alongside other models in informing subjective assessment by practitioners of the expected returns on assets in a multi-currency world.

REFERENCES

Adler, M & Dumas, B (1983). International portfolio choice and corporate finance: a synthesis, *Journal of Finance* **38**(3), 925–84

Dolde, W, Giaccotto, C, Mishra, DR & O'Brien, T (2011). Foreign exchange exposure and cost of equity for US companies: local versus global CAPM. *Journal of Applied Finance* **1**(1), 78–86

Elton, EJ & Gruber, MJ (1995). *Modern Portfolio Theory and Investment Analysis*. 5th edn., Wiley, New York

Grauer, FLA, Litzenberger, RH & Stehle, RE (1976). Sharing rules and equilibrium in an international capital market under uncertainty. *Journal of Financial Economics* **3**(3), 233–56

Jorion, P & Schwartz, E (1986) Integration vs. segmentation in the Canadian stock market. *Journal of Finance* **41**(3), 603–14

Karolyi, GA & Stulz, RM (2003). Are financial assets priced locally or globally? *Handbook of the Economics of Finance* **1**, 975–1020

Koedijk, KG, Kool, CJM, Schotman, PC & van Dijk, MA (2002). The cost of capital in international financial markets: local or global? *Journal of International Money and Finance* **21**(6), 905–29

Ng, DT (2004). The international CAPM when expected returns are time-varying. *Journal of International Money and Finance* **23**, 189–230

Sercu, P (1980). A generalization of the international asset pricing model. *Journal of Finance* **8**, 500–24

Solnik, BH (1974a). An equilibrium model of the international capital market. *Journal of Economic Theory* **8**, 500–24

Solnik, BH (1974b). The international pricing of risk: an empirical investigation of the world capital market structure. *Journal of Finance* **29**(2), 365–78

Stehle, R (1977). An empirical test of the alternative hypotheses of national and international pricing of risky assets. *Journal of Finance* **32**(2), 493–502

Stulz, RM (1981). International asset pricing. *Journal of Financial Economics* **9**, 383–406

Stulz, RM (1995a). International portfolio choice and asset pricing: an integrative survey. National Bureau of Economic Research, http://www.nber.org/papers/w4645

Stulz, RM (1995b). The cost of capital in internationally integrated markets: the case of Nestle. *European Financial Management* **1**(1), 11–22

Stulz, RM (1995c). Globalization of the capital markets and the cost of capital: the case of Nestle. *Journal of Applied Corporate Finance* **8**(3), 30–8

Wilkie, AD (unpublished). Why the capital asset pricing model fails in a multi-currency world. Proceedings of the 7th International AFIR Colloquium, 1997, 951–60